CERTAIN CLASSES OF UNIVALENT FUNCTIONS AND GENERALIZATIONS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

By

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To the memory of my

Loving Mother

CERTIFICATE

This is to certify that the work embodied in the thesis "Certain Classes of Univalent Functions and Generalizations of Functions with Bounded Boundary Rotation" by Surendra Prasad Dwivedi has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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September 1975 [Surendra Prasad Dwivedi]

CONTENTS

SYNOPSIS

The present thesis, which consists of five chapters is a study of the following classes of functions : Let m and M be arbitrary real numbers and $E = \{(m,M) : m > \frac{1}{2}, |m-1| < M \leq m\}$. $S(m,M)$ and $K(m,M)$ are the classes of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the disc $D = \{z : |z| < 1\}$ and satisfying there the conditions $\left|\frac{zf'(z)}{f(z)} - m\right| < M$ and $\left|1 + \frac{zf''(z)}{f'(z)} - m\right| < M$ respectively, where $(m,M) \in E$. Further $\Gamma(m,M)$ and $\Sigma(m,M)$ are the classes of functions $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$ analytic in the disc D punctured at the origin and satisfying the conditions $\left|\frac{zg'(z)}{g(z)} + m\right| < M$ and $\left|1 + \frac{zg''(z)}{g'(z)} + m\right| < M$ respectively, in D where $(m,M) \in E$. Further $V_\alpha(k,p)$ is the class of functions $h(z) = z + \sum_{n=1}^{\infty} C_{np+1} z^{np+1}$ analytic in D and satisfying there the condition $\int_0^{2\pi} \left|\mathrm{Re}\,\{e^{i\alpha}\,[1 + \frac{zh''(z)}{h'(z)}]\}\right| d\theta < k\pi \cos\alpha$, where $z = re^{i\theta}$, $0 \leq r < 1$, α is real with $|\alpha| < \pi/2$ and $k \geq 2$, p is a positive integer.

Chapter one, which is an introduction, describes definitions of various subclasses of analytic univalent, meromorphic univalent functions and functions with bounded boundary rotation. This chapter also describes the problems which have been investigated in the remaining four chapters.

Chapter two extends some results of S.D. Bernardi [Trans. Amer. Math. Soc. 135 (1969) 429-446] and S.K. Bajpai [Abstract 74T-B99, Notices Amer. Math. Soc. 21 (1974) A-376]. In fact Bernardi has proved that if $f(z)$ is analytic starlike with respect to origin (or convex) in D then $F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt$ is likewise starlike with respect to origin (or convex) in D, where c is a positive integer and Bajpai has proved that if $g(z)$ is meromorphic starlike (or meromorphic convex) in D then $G(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)\,dt$ is likewise meromorphic starlike (or meromorphic convex) in D where $c = 1$. Results analogous to those of Bernardi for the classes $S(m,M)$ and $K(m,M)$ when $c > -\frac{1-a}{1-b}$ and to those of Bajpai for the classes $\Gamma(m,M)$ and $\sum(m,M)$, when $c > \frac{a+b}{1-b}$, have bee obtained. In both cases $a = \frac{M^2-m^2+m}{M}$ and $b = \frac{m-1}{M}$. It is also proved that $F(z)$ is analytic starlike with respect to origin when $f(z)$ belongs to a class containing the class of starlike functions. Similar result for meromorphic starlike functions have also been derived.

In chapter three, using the method of T.H. MacGregor [J. London Math. Soc. (2) 9 (1975) 530-536], it is proved that if $f(z)$ belongs to $K(m,M)$ and $b \geq a$ then $\frac{zf'(z)}{f(z)}$ is subordinate to the function $G(z) = \frac{az(1-bz)^{-\frac{a+b}{b}}}{(1-bz)^{-a/b}-1}$. This in particular gives the bounds for $\left|\frac{f'(z)}{f(z)}\right|$ and the minimum values of m' and M' such that $f(z)$ belongs to $S(m',M')$.

In chapter four, converses and weak converses of some theorems proved in chapter two have been derived.

Chapter five is devoted to the study of the class $V_\alpha(k,p)$. Some representation theorems, distortion theorems, coefficient estimates and values of α for which functions of this class are univalent have been obtained. Distortion theorems yield radii of univalence, close-to-convexity and convexity as corollaries.

CHAPTER 1

INTRODUCTION

1.1 A function $f(z)$ is said to be univalent (Schlicht, simple or biuniform) in a domain D, if for any two points z_1 and z_2, $z_1 \neq z_2$, we have $f(z_1) \neq f(z_2)$. If $f(z)$ is univalent in D then so is the function $g(z) = \frac{f(z) - f(0)}{f'(0)}$, since $f'(0) \neq 0$. Hence normalization $f(0) = 0$, $f'(0) = 1$ of the univalent function is not an essential restriction. We shall denote by S the class of all functions $f(z)$ which are analytic and univalent in the open unit disc * $D = \{z : |z| < 1\}$ with the normalization $f(0) = 0$, $f'(0) = 1$. The Taylor expansion of such a function about the origin has the form

$$(1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The origin of the theory of univalent functions can be traced to a paper by P. Koebe in 1907 on the uniformization of algebraic curves [26] . In this paper Koebe proved that there is a constant K (called Koebe constant) such that the boundary of the map of D by any function $w = f(z)$ of the class S is always at a distance not less than K from the orig. Koebe's result soon attracted the attension of a number of eminent mathematicians [Plemelz [50], Gronwall[15],[16], Pick[48], Faber [13], Bieberbach [7]] .

*From here onwards we shall denote the unit disc $|z| < 1$ by D.

Gronwall [17] first gave the so called "area-principle" which asserts that if the function

(1.2) $$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

is univalent in D and regular there except at $z = 0$, where it has a simple pole, then

(1.3) $$\sum_{n=1}^{\infty} n|a_n|^2 \leq 1.$$

In 1916, Bieberbach [7] again proved the area principle and used it to obtain the precise value of K, namely $K = \frac{1}{4}$. This result can also be obtained from the results of Gronwall [17]. Bieberbach also proved that $|a_2| \leq 2$ for $f(z) \varepsilon S$. Since equality in this result is attained for the Koebe function

(1.4) $$f(z) = z(1 + \varepsilon z)^{-2}, \quad |\varepsilon| = 1$$

and $|a_n| = n$, $n = 2,3,4,\ldots.$ for this function it was conjectured by Bieberbach that for $f(z) \varepsilon S$,

(1.5) $$|a_n| \leq n.$$

In fact Bieberbach [7], Löwner [34], Garabedian and Schiffer [14], Pederson and Schiffer [47] and Pederson [46] have proved this conjecture for n = 2,3,4,5 and 6 respectively. Recently Ozawa and Kubota [43] have proved that

(1.6) $$\mathrm{Re}\,\{a_8\} \leq 8$$

if $1.9 \leq \text{Re}\{a_2\} \leq 2$ and $\left|\frac{\text{Im}\{a_2\}}{\text{Re}\{a_2\}}\right| \leq \frac{1}{20}$. Equality in (1.6) is attained for the Koebe function (1.4).

Various subclasses of univalent functions have been studied by different workers in this field. We give here the definitions of some important subclasses.

Definition A.1. A function $f(z)$ is said to be convex in a domain D if, whenever $w_1, w_2 \in f(D)$ (image of D under $f(z)$) then the straight line joining w_1 and w_2 is a subset of $f(D)$ i.e. $w_1 + t(w_2 - w_1) \in f(D)$, $0 \leq t \leq 1$. If, in addition $f(z) \in S$ then $f(z)$ is said to be a normalized convex univalent function. We shall denote by K, the class of all functions of S which are convex in D.

A necessary and sufficient condition for a function $f(z) \in S$ to be convex has been given by Robertson [54].

Theorem A.2. A function $f(z) \in S$ is convex in $|z| \leq r < 1$, if and only if

$$\text{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0 \quad \text{for } |z| \leq r < 1. \tag{1.7}$$

Robertson [54] has also defined the order α of a convex function $f(z) \in S$.

Definition A.3. A function $f(z) \in K$ is said to be of order α, $0 \leq \alpha < 1$ if

$$\text{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \alpha . \tag{1.8}$$

and for every real $\varepsilon > 0$, however small, there is a $z = z_0$, $|z_0| < 1$, for which

$$(1.9) \qquad \operatorname{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} < \alpha + \varepsilon .$$

We shall denote this class by $K(\alpha)$.

Definition B.1. A function $f(z)$ is said to be starlike with respect to $w_0 \varepsilon f(D)$ in a domain D if, whenever $w \varepsilon f(D)$ then the line joining w_0 and w is a subset of $f(D)$ i.e. $w_0 + t(w - w_0) \varepsilon f(D)$, $0 \leq t \leq 1$. If, in addition $f(z) \varepsilon S$ and $w_0 = 0$ then $f(z)$ is said to be a normalized starlike (with respect to origin) univalent function. We shall denote by S* the class of all functions of S which are starlike with respect to origin.

A necessary and sufficient condition for $f(z) \varepsilon S$ to be a member of S* is also due to Robertson [54].

Theorem B.2. A function $f(z) \varepsilon S$ is starlike with respect to origin in $|z| \leq r < 1$, if and only if,

$$(1.10) \qquad \operatorname{Re}\left\{z \frac{f'(z)}{f(z)}\right\} > 0 \quad \text{for } |z| \leq r .$$

As in the case of convex functions, the order α of starlike functions has also been defined by Robertson [54].

Definition B. . A function $f \varepsilon S^*$ is said to be starlike of order α, if

$$\operatorname{Re}\left\{z \frac{f'(z)}{f(z)}\right\} \geq \alpha \quad \text{for } |z| < 1, \qquad 0 \leq \alpha < 1$$

and if for every real $\varepsilon > 0$, however small, there is a z_0, $|z_0| < 1$, for which

$$\operatorname{Re}\left\{ z_0 \frac{f'(z_0)}{f(z_0)} \right\} < \alpha + \varepsilon$$

We shall denote this class by $S^*(\alpha)$.

It is clear from the above definitions that $K(\alpha) \subset S^*(\alpha)$ for every α, $0 \leq \alpha < 1$. We give here some important results of these two classes.

Theorem B.4.[40] A function $f \in K(\alpha)$, if and only if, $z f'(z) \in S^*(\alpha)$.

Theorem B.5.[37],[61] If $f \in K$, then $f \in S^*(\frac{1}{2})$.

Theorem B.6.[19][40] If $f \in S^*$ then $|a_n| \leq n$ and if $f \in K$ then $|a_n| \leq 1 < n$.

Yet other useful subclasses of $S^*(\alpha)$ and $K(\alpha)$ are due to Z.J. Jakubowski [20].

Definition C. $f \in S(m,M)$, if and only if,

$$\left| z \frac{f'(z)}{f(z)} - m \right| < M \quad \text{for } z \in D \text{ and } (m, M) \in E$$

where

$$E = \{(m, M) : m > \tfrac{1}{2}, \ |m-1| < M \leq m\}.$$

Definition D. $f \in K(m, M)$ if and only if

$$\left| 1 + z \frac{f''(z)}{f'(z)} - m \right| < M \quad \text{for } z \in D \text{ and } (m, M) \in E$$

where

$$E = \{(m, M) : m > \tfrac{1}{2}, \ |m-1| < M \leq m\}.$$

A class wider than the class of starlike functions is the class of spiral like functions introduced by L. Spacek [60].

Definition E.1 . A function $f \in S$ is spiral like in D if

$$\mathrm{Re}\ \{\xi z\,\frac{f'(z)}{f(z)}\} \geq 0, \quad z \in D$$

for some ξ such that $|\xi| = 1$.

Definition E.2. A function $f \in S$ is called α-spiral function if

$$\mathrm{Re}\ \{e^{i\alpha}\, z\,\frac{f'(z)}{f(z)}\} > 0\ , \quad z \in D.$$

We shall denote this class by $S(\alpha)$. Clearly $S(0) \equiv S^*$. The order β of univalent α-spiral functions has been introduced by Libera [32].

Definition E.3. A function $f \in S(\alpha)$ is said to be of order β in D, if

$$\mathrm{Re}\ \{e^{i\alpha}\ \frac{zf'(z)}{f(z)}\} > \beta \quad , \quad 0 \leq \beta < 1, \quad z \in D.$$

Another class, wider than the class of starlike functions is the class of close-to-convex univalent functions introduced W. Kaplan [24] in 1951.

Definition F.1. Let $f(z)$ be analytic in D, then $f(z)$ is close-to-convex for $|z| < 1$ if there exists a function $\phi(z) \in K$, such that

$$\mathrm{Re}\ \{\frac{f'(z}{\phi'(z}\} > 0 \qquad \text{for } z \in D.$$

We denote this class of functions by C.

Kaplan [24] further characterized close-to-convex functions, without reference to convex function ϕ in the following way.

Definition F.2. Let $f(z)$ be analytic and $f'(z) \neq 0$ in D. Then $f(z)$ is close-to-convex if and only, if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \{1 + z \frac{f''(z)}{f'(z)}\} \, d\theta > - \pi$$

where $\theta_1 < \theta_2$, $z = r e^{i\theta}$ and $r < 1$.

The order β and type λ for $f(z) \in C$ has been introduced by Libera [29].

Definition F._ : Let $f(z)$ be analytic in D with $f(0) = 0$, $f'(0) = 1$ and β, λ lie in the interval $[0, 1)$. Then $f(z)$ is said to be close-to-convex of order λ and type β, if and only if, there is some $F(z) \in S^*(\beta)$ and

$$\operatorname{Re} \{z \frac{f'(z)}{F(z)}\} \geq \lambda, \quad z \in D.$$

We denote this class of functions by $C(\lambda, \beta)$. The following results are well known.

Theorem F.4. [24] If $f(z) \in C$, then $f \in S$. Furthermore $S^* \subset C$.

Theorem F.5. [53] If $f \in C$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_n| \leq n$.

Definition G.1. Let $h(z)$ be analytic in D. We say that $h(z) \in P(\alpha)$ if $h(0) = 1$ and

$$\operatorname{Re} \{h(z)\} > \alpha, \quad z \in D, \quad 0 \leq \alpha < 1.$$

Theorem G.2 [40] If $h(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in P(0)$, then $|b_n| \leq 2$.

Definition H.1. Suppose $f(z)$ is analytic in D and $g(z)$ is univalent in D. Suppose further that $f(0) = g(0)$. We say that f is subordinate to g if $f(D) \subset g(D)$ and we write it as $f << g$.

Theorem H.2. [40] Suppose $f(z)$ is analytic in D and $g(z)$ is univalent in D and $f(0) = g(0)$. Then $f << g$ if and only if there exists a function $\phi(z)$ analytic in D, $|\phi(z)| \leq |z|$, such that $f(z) = g(\phi(z))$.

Theorem H.3. [37] If $f(z) \in S^*$, then $\frac{f(z)}{z} << \frac{1}{(1-z)^2}$.

Some analogous extensions ([23], [29], [35]) of the classes $K(\alpha)$, $S^*(\alpha)$ and $C(\alpha, \beta)$ etc. are carried over to the meromorphic univalent functions i.e. univalent functions which are analytic in the disc D except at the point $z = 0$ where the functions have simple pole.

Definition I.1. Let $f = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ be regular in $D_0 = \{z : 0 < |z| < 1\}$ is called meromorphic convex univalent function if compliment of f(D) is convex. We shall denote this class by $\sum$. Analytically, $f \in \sum$ if and only if

$$- \operatorname{Re} \{1 + z \frac{f''(z)}{f'(z)}\} > 0, \quad z \in D.$$

The class $\sum(\alpha)$ of meromorphic convex univalent functions of order α is defined as following.

Definition I.2. Let $f \in \sum$. Then f is meromorphic convex univalent of order α if and only if

$$-\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > \alpha, \qquad z \in D$$

and if for every real $\varepsilon > 0$, however small, there is a $z_o \in D$ such that

$$-\operatorname{Re}\left\{1 + \frac{z_o f''(z_o)}{f'(z_o)}\right\} < \alpha + \varepsilon.$$

Definition J.1. A function $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ analytic in $0 < |z| < 1$ is called meromorphic starlike with respect to origin if compliment of $f(D)$ is starlike with respect to origin. We shall denote this class by Γ^*. Analytically $f \in \Gamma^*$ if and only if

$$-\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > 0, \quad z \in D.$$

The class $\Gamma^*(\alpha)$ of meromorphic starlike (with respect to origin) function of order α is defined as following :

Definition J.2 : A function $f \in \Gamma^*$ is said to be of order α, if and only if

$$-\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \alpha, \quad z \in D, \quad 0 \leq \alpha < 1$$

and if for every $\varepsilon > 0$, however small, there is a $z_o \in D$ such that

$$-\operatorname{Re}\left\{z_o\frac{f'(z_o)}{f(z_o)}\right\} < \alpha + \varepsilon.$$

Theorem J. . $f \in \sum(\alpha)$ if and only if $-z f'(z) \in \Gamma^*(\alpha)$.

Theorem J.4.[12] If $f \in \Gamma^*$ and $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ then $|a_n| \leq \frac{2}{n+1}$.

Definition K. Denote by $B(\lambda,\beta)$, $0 \leq \beta$, $\beta < 1$, the family of functions $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ which are regular in $0 < |z| < 1$ and together with some $F \in \Gamma^*(\beta)$ such that

$$-\operatorname{Re}\left\{z \frac{f'(z)}{F(z)}\right\} > \lambda, \quad z \in D.$$

It may be noted [29] that $f \in B(\lambda,\beta)$ is not necessarily univalent.

Another subclass of analytic functions is the class of functions of bounded boundary rotation. This class of functions is not a subclass of S but it is closely related to the class of functions regular and univalent in D. This class is denoted by V_k. This class was introduced by Löwner [33].

Definition L.1 If $f(D)$ is a simple domain with a continuously differentiable boundary curve, the boundary rotation associated with $f(z)$ is defined to be the total variation of the direction angle made by the boundary tangent to $f(D)$ and the positive real axis as z makes a complete circuit of D. If the boundary of $f(D)$ is not so smooth let B be an arbitrary subdomain of $f(D)$ and L be a continuously differentiable closed curve in $f(D)$ containing B in its interior. Consider the greatest lower bound of the boundary rotation of all such curves L as B exhausts $f(D)$, this limit is defined as the boundary rotation associated with f.

Definition L.2. V_k is the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are locally univalent in D and boundary rotation of f(D) is atmost $k\pi$, $k \geq 2$.

Definition L.3. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k$ if $f'(z) \neq 0$ for $z \in D$ and

$$\int_0^{2\pi} |\mathrm{Re}\ \{1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\}|\ d\theta \leq k\pi$$

$z = re^{i\theta}$, $0 \leq r < 1$, $k \geq 2$.

Definition L.4. $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_\alpha^k$ if $f'(z) \neq 0$ in D and

$$\int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}[1 + z\frac{f''(z)}{f'(z)}]\}|\,d\theta \leq k.\pi \cos\alpha$$

$z = re^{i\theta}$, α real, $|\alpha| < \pi/2$, $k \geq 2$, $0 \leq r < 1$.

This class V_α^k is introduced and studied by E.J. Moulis [38].

Theorem L.5 [10]. If $f \in V_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ then $|a_n| \leq A_n$, where A_n is the coefficient of z^n in

$$f_k(z) = \frac{1}{k}\,[(\frac{1+z}{1-z})^{k/2} - 2].$$

Theorem L.6. [44] If $f \in V_k$, $2 \leq k \leq 4$ then f is univalent. In particular $V_2 \equiv k$.

1.2. Outlines of the thesis :

Usually the following type of problems are studied for univalent functions.

(1) Distortion theorems, i.e., determination of lower and upper bounds of $|f(z)|$, $|f'(z)|$ and $|z \frac{f'(z)}{f(z)}|$ etc.

(2) Coefficient estimates, when f belongs to various subclasses of univalent functions.

(3) Bounds for $\arg\{f(z)/z\}$ and $\arg\{f'(z)\}$.

(4) Radii of starlikeness and convexity of $f(z)$, and

(5) Certain transformations which are either class preserving or give some relation between two classes.

In the class of bounded boundary rotation, usually following types of problems are studied.

(a) Representation theorems.

(b) Distortion theorems.

(c) Coefficient estimates and

(d) Asymptotic coefficient estimates.

In the present work we have been mainly interested in problems of the type (4) and (5) for certain subclasses of S and meromorphic functions. We have also investigated problems of type (a), (b) and (c) for a subclass of V_k.

S.D. Bernardi [6] has proved that if $c = 1,2,3,\ldots$ then

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \in S^*(\alpha) \quad (\text{or } K(\alpha) \text{ or } C(\alpha, \beta)) \quad \text{for}$$

$\alpha = \beta = 0$ whenever $f \in S^*(\alpha)$ (or $K(\alpha)$ or $C(\alpha, \beta)$). S.K. Bajpai [2] has observed that this result is true if $c > -1$ and Bajpai and Srivastava [3] has observed that result is true for $0 \leq \alpha, \beta < 1$.

S.K. Bajpai [2] has also proved that if $g \in \Gamma^*(\alpha)$ (or $\Sigma(\alpha)$) then $G(z) = \frac{c}{z^{c+1}} \int_0^z t^c g(t)\,dt \in \Gamma^*(\alpha)$ (or $\Sigma(\alpha)$) if $c \geq 1$. In chapter two, we have proved similar results for some more subclasses of analytic, univalent and meromorphic univalent functions. We have also proved that $F \in S^*(\alpha)$ and $G \in \Gamma^*(\alpha)$ even in the cases when $f \notin S^*(\alpha)$ and $g \notin \Gamma^*(\alpha)$.

In Chapter three, we have proved that if $f \in K(m, M)$ then $z\frac{f'(z)}{f(z)} \ll \frac{az(1-bz)^{-(a+b)/b}}{(1-bz)^{-a/b}-1}$ provided $b \geq a$ where $b = \frac{m-1}{M}$ and $a = \frac{M^2-m^2+m}{M}$. Thus we get the bounds of $|z\frac{f'(z)}{f(z)}|$ and minimum of m_1 and M_1 so that $f \in K(m, M)$ implies $f \in S(m_1, M_1)$. This in particular contains the conjecture of Jack [22].

In chapter four, we have obtained converse and weak converse of some results proved in chapter two.

In last chapter, we have studied a subclass of the class V_α^k. Our subclass consists of those functions of V_α^k which are p-fold symmetric i.e., the functions having the expansion of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1} \quad , \; p = 1,2,3,\ldots$$

For this class we have determined representation theorems, distortion theorems coefficient estimates and value of α for which $f(z)$ becomes univalent. The distortion theorems give radii of univalence, close-to-convexity and convexity of functions of this class.

CHAPTER 2

SOME CLASSES OF UNIVALENT FUNCTIONS

2.1. Let m and M be arbitrary fixed real numbers which satisfy the relation $(m, M) \in E$ where

$$(2.1) \qquad E = \{(m, M), \ m > \tfrac{1}{2}, \ |m - 1| < M \leq m\}$$

Let us denote by $S(m, M)$ and $K(m, M)$ the class of functions of the form

$$(2.2) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular in the unit disc D and satisfying there the conditions

$$(2.3) \qquad \left| z \frac{f'(z)}{f(z)} - m \right| < M.$$

and

$$(2.4) \qquad \left| 1 + z \frac{f''(z)}{f'(z)} - m \right| < M$$

respectively, for $(m, M) \in E$.

Further let us denote by $\Gamma(m, M)$ and $\Sigma(m, M)$ the class of functions of the form

$$(2.5) \qquad g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$

regular in the disc D_0 and having a simple pole at origin and satisfying the conditions

$$(2.6) \qquad \left| z\,\frac{g'(z)}{g(z)} + m \right| < M$$

and

$$(2.7) \qquad \left| 1 + \frac{z\,g''(z)}{g'(z)} + m \right| < M$$

respectively, for $(m, M) \in E$.

If we take

$$(2.8) \qquad a = \frac{M^2 - m^2 + m}{M}$$

and

$$(2.9) \qquad b = \frac{m - 1}{M}$$

then the conditions (2.3), (2.4), (2.6) and (2.7) are equivalent to

$$(2.10) \qquad \frac{z\,f'(z)}{f(z)} = \frac{1 + a\,w_1(z)}{1 - b\,w_1(z)}$$

$$(2.11) \qquad 1 + \frac{z\,f''(z)}{f'(z)} = \frac{1 + a\,w_2(z)}{1 - b\,w_2(z)}$$

$$(2.12) \qquad \frac{z\,g'(z)}{g(z)} = -\frac{1 + a\,w_3(z)}{1 - b\,w_3(z)}$$

and

$$(2.13) \qquad 1 + \frac{z\,g''(z)}{g'(z)} = -\frac{1 + a\,w_4(z)}{1 - b\,w_4(z)}$$

respectively, for some $w_j(z)$, $j = 1,2,3,4$, regular and satisfying the conditions $w_j(0) = 0$, $|w_j(z)| < 1$ in D. In particular, if we choose

$a = \dfrac{\alpha - 2N\alpha + N}{N}$, $b = \dfrac{N-1}{N}$ and make $N \to \infty$ then (2.10), (2.11), (2.12) and (2.13) respectively imply that

(2.14) $$\operatorname{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha$$

(2.15) $$\operatorname{Re} \left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha$$

(2.16) $$\operatorname{Re} \left\{ \frac{z g'(z)}{g(z)} \right\} < -\alpha$$

and

(2.17) $$\operatorname{Re} \left\{1 + \frac{z g''(z)}{g'(z)}\right\} < -\alpha$$

But (2.14) implies that $f \in S^*(\alpha)$, (2.15) implies that $f \in K(\alpha)$, (2.16) implies that $g \in \Sigma^*(\alpha)$ and (2.17) implies that $g \in \Sigma(\alpha)$.

In 1964, M.S. Robertson [55] obtained a class preserving property. Functions $\dfrac{z}{(1-z)^2} \in S^*(0)$ and $\dfrac{z}{1-z} \in K(0)$ and are extremal in their respective classes for many purposes. Robertson proved that

$$\frac{2}{z} \int_0^z \frac{t}{(1-t)^2}\, dt \in S^*(0)$$

and

$$\frac{2}{z} \int_0^z \frac{t}{(1-t)}\, dt \in K(0)$$

and proposed the problem that if $f(z) \in S^*(0)$ (or $K(0)$) then whether $F(z) = \frac{2}{z} \int_0^z f(t)\,dt$ belongs to $S^*(0)$ (or $K(0)$) or not. In 1965 R.J. Libera [30] proved this fact. Then subsequently in 1969, S. D. Bernardi [6] extended this result and proved the following.

Theorem A [Bernardi] If $f(z) \in S^*(\alpha)$ (or $K(\alpha)$) then

$$\frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt \in S^*(\alpha) \quad (\text{or } K(\alpha)) \quad \text{for } c = 1,2,3,\dots \text{ and}$$

$\alpha = 0$.

Then in 1972, S.K. Bajpai and R.S.L. Srivastava [3] observed that the result is true for $\alpha (0 \le \alpha \le 1)$ and Bajpai [2] **observed that** result is true for $c > -1$.

Similar results for meromorphic starlike and meromorphic convex functions are due to S.K. Bajpai [2]. We state the result as a theorem.

Theorem B [Bajpai] If $g(z) \in \Gamma^*(\alpha)$ (or $\sum(\alpha)$) then

$$\frac{c}{z^{c+1}} \int_0^z t^c g(t)\,dt \in \Gamma^*(\alpha) \quad (\text{or } \textstyle\sum(\alpha)) \text{ for } 0 \le \alpha \le 1 \text{ and } c \ge 1.$$

In this chapter we shall extend above results for the classes $S(m, M)$, $K(m, M)$, $\Gamma(m, M)$ and $\sum(m, M)$. We also prove that functions defined in theorem A (and B) belongs to $S^*(\alpha)$ (and $\Gamma^*(\alpha)$) even if $f(z) \notin S^*(\alpha)$ $(g(z) \notin \Gamma^*(\alpha))$.

2.2. To prove our theorems we need the following lemma due to I.S. Jack [22].

Lemma 2.2.1. Suppose that $w(z)$ is analytic for $|z| \le r < 1$, $w(0) = 0$ and $|w(z_1)| = \max_{|z| = r} |w(z)|$ then $z_1 w'(z_1) = k\, w(z_1)$ where $k \ge 1$.

2.3. In this section we shall prove the following :

<u>Theorem 2.3.1</u>. <u>If</u> $f \in S(m, M)$ <u>and</u> $F(z)$ <u>is defined by</u>

$$(2.18) \qquad F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, \qquad c > -\frac{1-a}{1+b}$$

<u>then</u> $F \in S(m, M)$.

<u>Proof</u> : Let us choose a function $w(z)$ regular in D such that $w(0) = 0$ and

$$(2.19) \qquad z\,\frac{F'(z)}{F(z)} = \frac{1 + a\,w(z)}{1 - b\,w(z)} .$$

From (2.18) we get

$$[z^c F(z)]' = (c+1)\, z^{c-1} f(z)$$

or

$$(2.20) \qquad (c+1)\,\frac{f(z)}{F(z)} = c + z\,\frac{F'(z)}{F(z)}$$

From (2.19) and (2.20) we have

$$(2.21) \qquad (c+1)\,\frac{f(z)}{F(z)} = \frac{(1+c)+(a-bc)\,w(z)}{1 - b\,w(z)} .$$

Differentiating (2.21) logarithmically with respect to z and using (2.10) we get

$$(2.22) \qquad z\,\frac{f'(z)}{f(z)} - m = \frac{(1-m) + (a+bm)\,w(z)}{1 - b\,w(z)} + \frac{(a+b)\,z\,w'(z)}{\{1-b\,w(z)\}\{(1+c)+(a-bc)w(z)\}}$$

Now by lemma 2.2.1 for $|z| \leq r$ there is a point z_0 such that

$$(2.23) \qquad z_0\, w'(z_0) = k\, w(z_0)\,, \quad k \geq 1.$$

From (2.22) and (2.23) we have

$$(2.24)\qquad z_o \frac{f'(z_o)}{f(z_o)} - m = \frac{(1-m) + (a+bm)\,w(z_o)}{1 - b\,w(z_o)} + \frac{(a+b)\,k\,w(z_o)}{\{1 - b\,w(z_o)\}\{(1+c) + (a-bc)\,w(z_o)\}}$$

$$\equiv \frac{N(z_o)}{D(z_o)}$$

where

$$(2.25)\qquad N(z_o) = (1-m)(1+c) + [\,(1+c)\,(a+bm) + (a-bc)(1-m) + k(a+b)]\,w(z_o) + (a-bc)(a+bm)\,w^2(z_o).$$

and

$$(2.26)\qquad D(z_o) = (1+c) + (a-2bc-b)\,w(z_o) - b(a-bc)\,w^2(z_o)\ .$$

If we take

$$h = (1-m)\,(1+c)\ ,\ d = (1-m)\,(a-bc) + (1+c)(a+bm) + k(a+b),$$

$$e = (a-bc)(a+bm),\ j = (a-bc) - b(1+c) \text{ and } l = b(a-bc)$$

then

$$(2.27)\qquad N(z_o) = h + d\,w(z_o) + e\,w^2(z_o)$$

and

$$(2.28)\qquad D(z_o) = (1+c) + j\,w(z_o) - l\,w^2(z_o).$$

Now suppose that it were possible to have $M(r,w) = \max\limits_{|z|=r} |w(z)| = 1$ for some $r < 1$. At the point z_o where this occured we would have $|w(z)| = 1$ (but clearly $|w(z)| \neq 1$).Then

(2.29) $|N(z_o)|^2 = (h^2+d^2+e^2) + 2(e+h)\, d\, \mathrm{Re}\{w(z_o)\} + eh\, \mathrm{Re}\{w^2(z_o)\}$

and

(2.30) $|D(z_o)|^2 = (1+c)^2 + j^2 + l^2 + 2(1+c-l)j\, \mathrm{Re}\{w(z_o)\}$

$$- l(1+c)\, \mathrm{Re}\{w^2(z_o)\}$$

Now

(2.31) $|N(z_o)|^2 - M^2|D(z_o)|^2 = A + 2B\, \mathrm{Re}\{w(z_o)\} + C\mathrm{Re}\{w^2(z_o)\}$

where

$$\begin{aligned}
A &= (h^2+d^2+e^2) - M^2\{(1+c)^2 + j^2 + l^2\} \\
&= [(1-m)^2(1+c)^2+(1-m)^2(a-bc)^2+(1+c)^2(a+bm)^2+k^2(a+b)^2 \\
&\quad + 2(1-m)(a-bc)(1+c)(a+bm) + 2(1+c)(a+bm)\, k(a+b) \\
&\quad + 2(1-m)(a-bc)k(a+b) + (a-bc)^2(a+bm)^2] - [(1+c)^2 \\
&\quad + (a-bc)^2 + b^2(1+c)^2 - 2b(a-bc)(1+c) + b^2(a-bc)^2]M^2 \\
&= [M^2(1+c)^2b^2 + M^2b^2(a-bc)^2 + (1+c)^2M^2 + k^2(a+b)^2 \\
&\quad -2M^2b(a-bc)(1+c) + 2Mk(1+c)(a+b) - 2Mb(a-bc)k(a+b) \\
&\quad + (a-bc)^2M^2] - [(1+c)^2 M^2 + (a-bc)^2 M^2 + M^2b^2(1+c)^2 \\
&\quad - 2M^2b(a-bc)(1+c) + M^2b^2(a-bc)^2] \\
&= k(a+b)\, [k(a+b) + 2M(1+c) - 2Mb(a-bc)]
\end{aligned}$$

$$\begin{aligned}
B &= (e+h)d - M^2 j\,(1+c-l) \\
&= [(a-bc)(a+bm) + (1-m)(1+c)][(1-m)(a-bc) + (1+c)(a+bm) \\
&\quad + k(a+b)] - M^2[(a-bc) - b(1+c)] \times [(1+c) - b(a-bc)]
\end{aligned}$$

$$= M\{(a-bc) - b(1+c)\}\{-Mb(a-bc) + M(1+c) + k(a+b)\}$$

$$-M^2\{(a-bc) - b(1+c)\}\{(1+c) - b(a-bc)\}$$

$$= \{(a-bc) - b(1+c)\}\{-M^2b(a-bc) + M^2(1+c) + Mk(a+b)$$

$$- M^2(1+c) + M^2 b(a-bc)\}$$

$$= Mk(a+b)\{(a-bc) - b(1+c)\}$$

and

$$C = eh + M^2 l\,(1+c)$$

$$= (1-m)\,(1+c)\,(a-bc)(a+bm) + M^2b(a-bc)\,(1+c)$$

$$= -M^2b(1+c)(a-bc) + M^2b(a-bc)(1+c)$$

$$= 0.$$

Since $C = 0$ from (2.31) it is clear that

(2.32) $\quad |N(z_0)|^2 - M^2|D(z_0)|^2 \geq 0 \quad$ provided $A \pm 2B \geq 0.$

Now

$$A + 2B = k(a+b)\,[k(a+b) + 2M(1+c) - 2Mb(a-bc) + 2M(a-bc) - 2Mb(1+c)]$$

$$= k(a+b)\,[k(a+b) + 2M\{(1+c) - b(a-bc) + (a-bc) - b(1+c)\}]$$

$$= k(a+b)\,[k(a+b) + 2M(1-b)\{(1+a) + c(1-b)\}]$$

$$\geq 0\,.$$

$$A - 2B = k(a+b)\,[k(a+b) + 2M(1+c) - 2Mb(a-bc) - 2M(a-bc) + 2Mb(1+c)]$$

$$= k(a+b)[k(a+b) + 2M\{(1+c) - b(a-bc) - (a-bc) + b(1+c)\}]$$

$$= k(a+b)[k(a+b)+2M(1+b)\,\{(1-a) + c(1+b)\}]$$

$$\geq 0.$$

Thus we have proved (2.32) which along with (2.24) gives

$$\left| z_0 \frac{f'(z_0)}{f(z_0)} - m \right| \geq M.$$

But this is a contradiction to the fact $f \varepsilon S(m, M)$. So we can not have $M(r,w) = 1$. Since this is true for every $r < 1$ and since $M(0, w) = 0$ it is clear that we must have $M(r, w) < 1$ and hence $|w(z)| < 1$ for $|z| < 1$. Therefore, $F \varepsilon S(m, M)$ follows from (2.19).

<u>Corollary 2.3.1</u>. <u>If</u> $f \varepsilon K(m,M)$ <u>and</u> F <u>is defined by</u> (2.18) <u>then</u> $F \varepsilon K(m, M)$. provided $c \geq -\frac{1-a}{1+b}$

<u>Proof</u> : We can write (2.18) in the form

$$z F'(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} . t f'(t) dt.$$

Since $f \varepsilon K(m, M)$ it is easy to see that $z f'(z) \varepsilon S(m, M)$. Therefore by theorem 2.3.1. we get $z F'(z) \varepsilon S(m, M)$ which implies that $F(z) \varepsilon K(m, M)$.

<u>Remark 1</u>. In theorem 2.3.1 if we put

(i) $m = M$ and $m \to \infty$ then results of Bernardi [6] follow .

(ii) $m = \frac{\alpha - 2\alpha N + N}{N}$ and $M = \frac{N-1}{N}$ and $N \to \infty$ then results of Bajpai [2] follow.

(iii) $m = M$, $c = 1$ and $m \to \infty$ then results of Libera[30] follow.

<u>Theorem 2.3.2</u>. <u>If</u> $f \varepsilon S^*(\alpha)$ <u>and</u> $g \varepsilon S(m, M)$ <u>and</u> $F(z)$ <u>is defined by</u>

$$(2.33) \qquad F(z) = \frac{c+2}{z^{c+1}} \int_0^z t^{c-1} f(t) g(t) dt, \qquad c \geq 0$$

<u>then</u> $F \in S^*(\alpha)$ <u>if</u>

$$(m,M) \in \{(m,M): m \geq \frac{4c+3+5\alpha}{4(c+1+\alpha)}, \ |m-1| < M \leq (m-1) + \frac{1-\alpha}{2(c+1+\alpha)}\}$$

<u>Proof</u> : It can be easily seen that

$$|m-1| \leq (m-1) + \frac{1-\alpha}{2(c+1+\alpha)}$$

only if

$$m \geq \frac{4c+3+5\alpha}{4(c+1+\alpha)} .$$

so it is sufficient to prove that $F \in S^*(\alpha)$ provided

$$M \leq (m-1) + \frac{1-\alpha}{2(c+1+\alpha)} .$$

From (2.33) we have

$$z^{c+1} F'(z) + (c+1)z^c F(z) = (c+2) z^{c-1} f(z) g(z)$$

or

$$(2.34) \qquad \frac{zF'(z)}{F(z)} + (c+1) = (c+2) \frac{f(z) g(z)}{zF(z)} .$$

Let us choose $w(z)$ regular in D and satisfying there the condition

$$(2.35) \qquad \frac{zF'(z)}{F(z)} = \frac{1+(2\alpha-1) w(z)}{1+w(z)} .$$

It is clear from (2.35) that $w(0) = 0$. From (2.34) and (2.35) we have

$$(2.36) \quad (c+2) \frac{f(z)g(z)}{zF(z)} = (c+1) + \frac{1+(2\alpha-1) w(z)}{1+w(z)}$$

$$= (c+2) \cdot \frac{1+\frac{c+2\alpha}{c+2} w(z)}{1+w(z)}$$

Differentiating (2.36) logarithmically with respect to z and using (2.35) we get

$$z \frac{f'(z)}{f(z)} = 1 - \frac{zg'(z)}{g(z)} + \frac{1+(2\alpha-1) w(z)}{1+w(z)} + \frac{c+2\alpha}{c+2} \cdot \frac{z w'(z)}{1+\frac{c+2\alpha}{c+2} w(z)}$$

$$- \frac{z w'(z)}{1+w(z)}$$

$$(2.37)\quad \frac{z\,f'(z)}{f(z)} = (1-m) - \left(\frac{z\,g'(z)}{g(z)} - m\right) + \frac{1+(2\alpha-1)\,w(z)}{1+w(z)}$$

$$- \frac{2(1-\alpha)}{c+2} \cdot \frac{z\,w'(z)}{(1+w(z))\left(1+\frac{c+2\alpha}{c+2}w(z)\right)} .$$

Now by Jacks lemma 2.3.1 for $|z| \le r$ there is a point z_0 such that

$$(2.38)\qquad z_0 w'(z_0) = k\,w(z_0)\,, \quad k \ge 1.$$ From this lemma and (2.37) we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = (1-m) - \left(\frac{z_0 g'(z_0)}{g(z_0)} - m\right) + \frac{1+(2\alpha-1)\,w(z_0)}{1+w(z_0)}$$

$$- \frac{2(1-\alpha)}{c+2} \cdot \frac{k\,w(z_0)}{(1+w(z_0))\left(1+\frac{c+2\alpha}{c+2}\,w(z_0)\right)}$$

or

$$(2.39)\qquad \operatorname{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \le (1-m) + \left|\frac{z_0\,g'(z_0)}{g(z_0)} - m\right|$$

$$+ \operatorname{Re}\left\{\frac{\{1+(2\alpha-1)\,w(z_0)\}\{1+\overline{w}(z_0)\}}{|1+w(z_0)|^2}\right\}$$

$$- \frac{2(1-\alpha)}{c+2} \cdot \operatorname{Re}\left\{\frac{k\,w(z_0)(1+\overline{w(z_0)})(1+\frac{c+2\alpha}{c+2}\overline{w(z_0)})}{|1+w(z_0)|^2\,\left|1+\frac{c+2\alpha}{c+2}\,w(z_0)\right|^2}\right\}$$

$$\le (M-m+1) + \frac{1+2\alpha\operatorname{Re} w(z_0) + (2\alpha-1)\,|w(z_0)|^2}{1+2\operatorname{Re} w(z_0) + |w(z_0)|^2}$$

$$- \frac{2(1-\alpha)k}{c+2}\;\frac{\operatorname{Re}\left\{w(z_0) + \frac{2(c+1+\alpha)}{c+2}|w(z_0)|^2 + \frac{c+2\alpha}{c+2}\overline{w(z_0)}\,|w(z_0)|^2\right\}}{(1+2\operatorname{Re} w(z_0)+|w(z_0)|^2)\left(1+\frac{2(c+2\alpha)}{c+2}\operatorname{Re} w(z_0)+\left(\frac{c+2\alpha}{c+2}\right)^2|w(z_0)|^2\right)}$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some $r < 1$. At the point $w(z)$ where this occured we would have $|w(z)| = 1$ then from (2.39) we have

$$\mathrm{Re}\left\{ \frac{z_0 \cdot f'(z_0)}{f(z_0)} \right\} \leq \alpha + (M - m + 1) - \frac{2(1-\alpha)(c+1+\alpha)\,k}{(c+2)^2 + 2(c+2)(c+2\alpha)\mathrm{Re}\, w(z_0) + (c+2\alpha)^2}$$

$$\leq \alpha + (M - m + 1) - \frac{2(1-\alpha)(c+1+\alpha)}{(c+2)^2 + 2(c+2)(c+2\alpha)\mathrm{Re}\, w(z_0) + (c+2\alpha)^2}$$

$$= \alpha + \frac{(M-m+1)\{(c+2)^2 + 2(c+2)(c+2\alpha)\,\mathrm{Re}\, w(z_0) + (c+2\alpha)^2\} - 2(1-\alpha)(c+1+\alpha)}{(c+2)^2 + 2(c+2)(c+2\alpha)\,\mathrm{Re}\, w(z_0) + (c+2\alpha)^2}$$

$$< \alpha + \frac{2(c+1+\alpha\;\{2(c+1+\alpha)(M-m+1) - (1-\alpha)\}}{(c+2)^2 + 2(c+2)(c+2\alpha)\mathrm{Re}\, w(z_0) + (c+2\alpha)^2}$$

$$< \alpha$$

provided $\quad M \leq \dfrac{2(m-1)(c+1+\alpha) + (1-\alpha)}{2c+1+\alpha)}$

But this is contradiction to the fact that $f \in S^*(\alpha)$. So we can not have $M(r, w) \neq 1$. Since $M(0, w) = 0$ and $M(r, w) \neq 1$ for every $r < 1$ so $|w(z)| < 1$ and therefore from (2.35) we have $F \in S^*(\alpha)$.

Remark 2. Let us take $G(z) = \dfrac{f(z)\,g(z)}{z}$ then (2.33) reduces to

$$F(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c\, G(t)\, dt.$$

Bernardi [6] proved that $F(z) \varepsilon S^*(0)$ if $G(z) \varepsilon S^*(0)$. If we take $f(z)$ and $g(z)$ such that

$$z \frac{f'(z)}{f(z)} = \frac{1-z}{1+z}$$

and

$$z \frac{g'(z)}{g(z)} = 1 - \frac{z}{2(c+1)}$$

then $f \varepsilon S^*(0)$ and $g(z) \varepsilon S(1, \frac{1}{2(c+1)})$ and

$$z \frac{G'(z)}{G(z)} = \frac{2(c+1) - (2c+3)z - z^2}{2(c+1)(1+z)}$$

If we take z real and between $\frac{\sqrt{4c^2+20c+17} - (2c+3)}{2}$ and 1 then it is easily seen that $\mathrm{Re}\{z \frac{G'(z)}{G(z)}\} < 0$ and hence $G(z) \notin S^*(0)$. But by theorem 2.3.2 we have $F(z) \varepsilon S^*(\alpha)$.

<u>Theorem 2.3.3</u>. <u>Let</u> $f \varepsilon \Gamma(m, M)$ <u>and</u> $F(z)$ <u>be defined by</u>

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt \tag{2.40}$$

<u>then</u> $F \varepsilon \Gamma(m, M)$ <u>if</u> $c \geq \max. \{\frac{a+b}{1-b}, 1\}$

<u>Proof</u> : From (2.40) we have

$$z^{c+1} F'(z) + (c+1) z^c F(z) = c z^c f(z)$$

or

$$(2.41) \qquad z\frac{F'(z)}{F(z)} + c + 1 = c.\frac{f(z)}{F(z)} .$$

Let us choose a regular function $w(z)$ such that

$$(2.42) \qquad z\frac{F'(z)}{F(z)} = -\frac{1 + a\,w(z)}{1 - b\,w(z)} .$$

It is clear from (2.42) that $w(0) = 0$. By (2.41) and (2.42) we have

$$(2.43) \qquad \frac{c\,f(z)}{F(z)} = \frac{c - (a + b + bc)\,w(z)}{1 - b\,w(z)}$$

Differentiating (2.43) logarithmically with respect to z and using (2.42) we get

$$z\frac{f'(z)}{f(z)} = -\frac{1 + aw(z)}{1 - bw(z)} - \frac{a + b + bc}{c} \cdot \frac{z\,w'(z)}{1 - \frac{a+b+bc}{c}\,w(z)} + \frac{bz\,w'(z)}{1 - bw(z)}$$

or

$$(2.44) \qquad \frac{zf'(z)}{f(z)} + m = m - \frac{1+a\,w(z)}{1-b\,w(z)} - \frac{a+b}{c}\,\frac{z\,w'(z)}{(1-b\,w(z))(1-\frac{a+b+bc}{c}\,w(z))}$$

$$= \frac{c(m-1) - \{(m-1)(a+b+bc)+ c(a+bm)\}\,w(z)+(a+bm)(a+b+bc)w^2(z) - (a+b)\,z\,w'(z)}{\{1 - b\,w(z)\}\,\{1 - \frac{a + b + bc}{c}\,w(z)\}}$$

Now by Jack's lemma 2.3.1 for $|z| \leq r$ there is a point z_o such that $z_o w'(z_o) = k\, w(z_o)$, $k \geq 1$. Using this in (2.44) we have

$$\text{(2.45)} \qquad \frac{z_o f'(z_o)}{f(z_o)} + m = \frac{c(m-1) - \{(m-1)(a+b+bc) + c(a+bm) + k(a+b)\} w(z_o) + (a+bm)(a+b+bc)\, w^2(z_o)}{(1 - b\, w(z_o))\, (c - (a+b+bc)\, w(z_o)).}$$

$$\equiv \frac{N(z_o)}{D(z_o)}$$

where

$$N(z_o) = c(m-1) - \{(m-1)(a+b+bc) + c(a+bm) + k(a+b)\}\, w(z_o) + (a+bm)(a+b+bc)\, w^2(z_o)$$

and

$$D(z_o) = c - (a + b + 2bc)\, w(z_o) + b\,(a + b + bc)\, w^2(z_o).$$

Now suppose it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some $r < 1$. At the point $w(z)$ where this occured we would have $|w(z)| = 1$. Then we have

$$\begin{aligned}
\text{(2.46)} \quad |N(z_o)|^2 = {} & M^2b^2c^2 + \{Mb(a+b+bc) + Mc + k(a+b)\}^2 + M^2(a+b+bc)^2 \\
& - 2\,[Mbc\,\{Mb(a+b+bc) + Mc + k(a+b)\} + M(a+b+bc) \\
& \{Mb(a+b+bc) + Mc + k(a+b)\}]\ \mathrm{Re}\ w(z_o) \\
& + 2M^2bc\,(a+b+bc)\ \mathrm{Re}\ w^2(z_o). \\
= {} & M^2b^2c^2 + \{Mb(a+b+bc) + Mc + k(a+b)\}^2 + M^2(a+b+bc)^2 \\
& - 2\,[\{Mb(a+b+bc) + Mc + k(a+b)\}\,\{a+b+2bc\}\,M]\ \mathrm{Re}\ w(z_o) \\
& + 2M^2bc(a+b+bc)\ \mathrm{Re}\ w^2(z_o).
\end{aligned}$$

and

$$(2.47) \quad |D(z_o)|^2 = c^2 + (a+b+2bc)^2 + b^2(a+b+bc)^2$$
$$- 2(a+b+2bc)\ \{c + b\,(a+b+bc)\}\ \mathrm{Re}\ w(z_o)$$
$$+ 2bc\,(a+b+bc)\ \mathrm{Re}\ w^2(z_o)$$

From (2.46) and (2.47) we have

$$(2.48) \quad |N(z_o)|^2 - M^2|D(z_o)|^2 = A - 2B\ \mathrm{Re}\ w(z_o) + C\ \mathrm{Re}\ w^2(z_o)$$

where

$$A = M^2b^2c^2 + M^2b^2(a+b+bc)^2 + M^2c^2 + k^2(a+b)^2 + 2M^2bc(a+b+bc)$$
$$+ 2Mck\,(a+b) + (a+b+bc)\ 2Mbk(a+b) + M^2(a+b+bc)^2$$
$$- M^2c^2 - M^2(a+b+2bc)^2 - M^2b^2(a+b+bc)^2$$
$$= k(a+b)\,[k(a+b) + 2Mc + 2Mb\,(a+b+bc)]$$

$$B = \{Mb(a+b+bc) + Mc + k(a+b)\}M(a+b+2bc)$$
$$- (a+b+2bc)\{c + b(a+b+bc)\}M^2$$
$$= Mk(a+b)\,(a+b+2bc)$$

and

$$C = 2M^2bc(a+b+cb) - 2M^2bc(a+b+bc)$$
$$= 0\ .$$

Since $C = 0$ it is easy to see from (2.48) that

$$(2.49) \quad |N(z_o)|^2 - M^2|D(z_o)|^2 \geq 0 \ \text{ provided }\ A \pm 2B \geq 0.$$

Now

$$A - 2B = k(a + b)\,[k(a + b) + 2M\,\{c + ab + b^2 + b^2c - a - b - 2bc\}]$$

$$= k(a + b)\,[k(a + b) + 2M\,\{c(1-b) - bc(1-b) - a(1-b) - b(1-b)\}]$$

$$= k(a + b)\,[k(a + b) + 2M(1-b)\,(c - bc - a - b)]$$

$$\geq 0\,, \quad \text{provided} \quad c \geq \frac{a + b}{1 - b}\,.$$

and

$$A + 2B = k(a + b)\,[k(a + b) + 2M\,\{c + ab + b^2 + b^2c + a + b + 2bc\}\,]$$

$$= k(a + b)\,[k(a + b) + 2M(1 + b)\,\{c(1 + b) + (a + b)\}]$$

$$\geq 0\,.$$

Thus we see that (2.49) holds. (2.49) alongwith (2.45) gives

$$\left|\, z_0\,\frac{f'(z_0)}{f(z_0)} + m \,\right| \geq M.$$

But this is contradiction to the fact that $f \in \Gamma(m, M)$. Hence $M(r,w) \neq 1$ for every $r < 1$. Since $M(0,w) = 0$ and $M(r, w) \neq 1$ we have $M(r,w) < 1$ and hence $|w(z)| < 1$ for $|z| < 1$. This alongwith (2.42) gives $F \in \Gamma(m, M)$. This completes the proof of the theorem.

<u>Corollary 2.3.2</u>. <u>If</u> $f \in \sum(m, M)$ <u>and</u> $F(z)$ <u>is defined by</u> (2.40) <u>then</u> $F \in \sum(m, M)$. Provided $c \geq \max\{\frac{a+b}{1-b}, 1\}$

<u>Proof</u>. We can write (2.40) as

$$z\,F'(z) = \frac{c}{z^{c+1}} \int_0^z t^c.\, t\, f'(t)\, dt.$$

Since $f \in \sum(m, M)$ we have $z\,f'(z) \in \sum(m, M)$ and hence from theorem 2.3.3 we get $z\,F'(z) \in \sum(m, M)$. So $F(z) \in \sum(m, M)$.

Remark 3. If we take $m = M$ and $m \to \infty$ then results of Bajpai [2] follow from theorem 2.3.3 and Corollary 2.3.2.

Theorem 2.3.4. Let $f \in \Gamma^*(\alpha)$ and $g \in \Gamma(m,M)$ and $F(z)$ be defined by

$$(2.50) \qquad F(z) = \frac{c}{z^{c+1}} \int_0^z t^{c+1} f(t)\, g(t)\, dt, \qquad c \geq 1$$

then $F \in \Gamma^*(\alpha)$ provided

$$(m,M) \in \{(m,M) : m \geq \frac{4c+3(1-\alpha)}{4(c+1-\alpha)},\ |m-1| < M \leq (m-1) + \frac{1-\alpha}{2(c+1-\alpha)}\}.$$

Proof : It can be easily seen that

$$|m-1| \leq (m-1) + \frac{1-\alpha}{2(c+1-\alpha)}$$

only if

$$m \geq \frac{4c+3(1-\alpha)}{4(c+1-\alpha)},$$

so it is sufficient to prove that $F \in \Gamma^*(\alpha)$ provided

$$M \leq (m-1) + \frac{1-\alpha}{2(c+1-\alpha)}.$$

From (2.50) we have

$$z^{c+1} F'(z) + (c+1) z^c F(z) = c\, z^{c+1} f(z)\, g(z)$$

or

$$(2.51) \qquad \frac{czf(z)\, g(z)}{F(z)} = c + 1 + \frac{zF'(z)}{F(z)}.$$

Let us choose a regular function $w(z)$ such that

$$(2.52) \qquad \frac{zF'(z)}{F(z)} = -\frac{1 + (2\alpha-1)\, w(z)}{1 + w(z)}.$$

It is clear from (2.52) that $w(0) = 0$. From (2.51) and (2.52) we have

$$(2.53) \qquad \frac{c\, zf(z)\, g(z)}{F(z)} = c + 1 - \frac{1 + (2\alpha-1)\, w(z)}{1 + w(z)}$$

$$= \frac{c + \{c + 2(1-\alpha)\}\, w(z)}{1 + w(z)}$$

Differentiating (2.53) logarithmically with respect to z and using (2.52) we have

$$(2.54) \qquad \frac{z f'(z)}{f(z)} = -1 - \frac{z g'(z)}{g(z)} - \frac{1 + (2\alpha-1)\, w(z)}{1 + w(z)}$$

$$+ \frac{\{c + 2(1-\alpha)\}\, z w'(z)}{c + \{c + 2(1-\alpha)\} w(z)} - \frac{z w'(z)}{1 + w(z)}$$

$$= -1 + m - \left(\frac{z g'(z)}{g(z)} + m\right) - \frac{1 + (2\alpha-1)\, w(z)}{1 + w(z)}$$

$$+ \frac{2(1-\alpha)\, z w'(z)}{(1 + w(z))\{c + (c + 2 - 2\alpha) w(z)\}}$$

By Jack's lemma 2.2.1 for $|z| \leq r$ there is a point z_0 such that $z_0 w'(z_0) = k\, w(z_0)$, $k \quad 1$. Using this lemma in (2.54) we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = (m - 1) - \left(\frac{z g'(z_0)}{g(z_0)} + m\right) - \frac{1 + (2\alpha-1)\, w(z_0)}{1 + w(z_0)}$$

$$+ \frac{2(1-\alpha)\, k\, w(z_0)}{(1 + w(z_0))\, \{c + (c + 2 - 2\alpha)\, w(z_0)\}}.$$

Now suppose it were possible to have $M(r, w) = \max\limits_{|z| = r} |w(z)| = 1$ for some $r < 1$. At the point $w(z)$ where this occured we would have $|w(z)| = 1$. Then we have

$$\operatorname{Re} \left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \geq (m - 1) - \left|\frac{z g'(z_0)}{g(z_0)} + m\right| - \operatorname{Re} \frac{1+(2\alpha-1)\, w(z_0)}{1 + w(z_0)}$$

$$+ \operatorname{Re} \frac{2(1-\alpha)\, k\, w(z_0)}{(1+ w(z_0))\, \{c + (c+2 - 2\alpha) w(z_0)\}}$$

$$\geq -(M - m + 1) - \alpha + \frac{2k(1-\alpha)(c + 1 - \alpha)}{c^2 + 2c(c+2-2\alpha) \operatorname{Re} w(z_0) + (c+2-2\alpha)^2}$$

$$\geq -\alpha + \frac{2(1-\alpha)(c+1-\alpha) - (M-m+1)\{c^2 + 2c(c+2-2\alpha)\operatorname{Re} w(z_0) + (c+2-2\alpha)^2\}}{c^2 + 2c(c+2-2\alpha)\operatorname{Re} w(z_0) + (c+2-2\alpha)^2}$$

$$> -\alpha + \frac{2(1-\alpha)(c+1-\alpha) - (M-m+1)\{c+c+2-2\alpha\}^2}{c^2 + 2c(c+2-2\alpha)\operatorname{Re} w(z_0) + (c+2-2\alpha)^2}$$

$$= -\alpha + \frac{2(c+1-\alpha)\{(1-\alpha) - 2(M-m+1)(c+1-\alpha)\}}{c^2 + 2c(c+2-2\alpha)\operatorname{Re} w(z_0) + (c+2-2\alpha)^2}$$

$$\geq -\alpha \quad \text{provided} \quad M \leq \frac{2(m-1)(c+1-\alpha) + (1-\alpha)}{2(c+1-\alpha)} .$$

This contradicts the fact $f \in \Gamma^*(\alpha)$ hence $M(r, w) \neq 1$ for every $r < 1$ so $M(r, w) < 1$ because $M(0, w) = 0$. Hence $|w(z)| < 1$ for $|z| < 1$. This alongwith (2.52) gives $F \in \Gamma^*(\alpha)$. This completes the proof of the theorem.

Remark 1. Let us take $G(z) = z f(z) g(z)$ then (2.50) reduces to

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c G(t)\, dt.$$

Bajpai [2] has proved that $F \in \Gamma^*(0)$ if $G(z) \in \Gamma^*(0)$. If we take $f(z)$ and $g(z)$ such that

$$z\frac{f'(z)}{f(z)} = -\frac{1-z}{1+z}$$

and
$$z\frac{g'(z)}{g(z)} = -1 + \frac{z}{2(c+1)}$$

then $f(z) \in \Gamma^*(0)$ and $g(z) \in \Gamma(m, M)$ for $m = 1$ and $M = \frac{1}{2(c+1)}$.

But

$$z\,\frac{G'(z)}{G(z)} = -\frac{1-z}{1+z} - 1 + \frac{z}{2(c+1} + 1$$

$$= -\frac{2(c+1) - (2c+3)z - z^3}{2(c+1 \quad z}$$

If we take z real and between $\dfrac{\sqrt{4c^2 + 20c + 17} - (2c+3)}{2}$ and 1 then it is easily seen that $\operatorname{Re} z\,\dfrac{G'(z)}{G(z)} > 0$ and hence $G(z) \notin \Gamma^*(0)$. But by theorem 2.3.4 we have $F \in \Gamma^*(\alpha)$.

CHAPTER 3

A SUBORDINATION TO A CERTAIN CLASS OF ANALYTIC FUNCTIONS

3.1 It is well known that convex functions are starlike with respect to origin. In 1933 A. Marx [37] and E. Strohacker [61] proved that if $f(z) \in K(0)$ then $f(z) \in S^*(\beta)$ where $\beta \geq \frac{1}{2}$. This result is sharp as it can be seen from the function $z/(1-z)$. In 1971, I.S. Jack [22] generalized this result and proved the following.

<u>Theorem C</u> [Jack] <u>If</u> $f(z) \in K(\alpha)$ <u>then</u> $f(z) \in S^*(\beta(\alpha))$ <u>where</u>

$$(3.1) \qquad \beta(\alpha) \geq \frac{(2\alpha-1) - \sqrt{9 - 4\alpha + 4\alpha^2}}{4}$$

This bound for $\beta(\alpha)$ is not sharp. In many cases the function

$$(3.2) \qquad A(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \text{if } 0 \leq \alpha \leq 1 \text{ and } \alpha \neq \frac{1}{2} \\ -\log(1-z) & \text{if } \alpha = \frac{1}{2} \end{cases}$$

is extremal for the class of convex functions and for this function

$$(3.3) \qquad \operatorname{Re}\left\{ z\frac{A'(z)}{A(z)} \right\} > \begin{cases} \dfrac{1-2\alpha}{4^{1-\alpha}(1-2^{2\alpha-1})} & \text{if } \alpha \neq \frac{1}{2} \\ \dfrac{1}{\log 4} & \text{if } \alpha = \frac{1}{2} \end{cases}$$

So Jack conjectured that

$$(3.4) \qquad \beta(\alpha) \geq \begin{cases} \dfrac{1-2\alpha)}{4^{1-\alpha}(1-2^{2\alpha}-1)} & \text{if } \alpha \neq \frac{1}{2} \\ \dfrac{1}{\log 4} & \text{if } \alpha = \frac{1}{2} \end{cases}$$

Recently T. H. MacGregor [36] has settled this conjecture. MacGregor's proof is very nice and independent of any classical result. MacGregor's result is as following :

Theorem D [MacGregor] If f is convex of order α i.e. $f \in K(\alpha)$ then $z\dfrac{f'(z)}{f(z)} \prec J(z)$ where

$$(3.5) \qquad J(z) = \begin{cases} \dfrac{(2\alpha-1)z}{(1-z)^{2(1-\alpha)}\{1-(1-z)^{2\alpha-1}\}} & \text{if } \alpha \neq \frac{1}{2} \\ -\dfrac{z}{(1-z)\log(1-z)} & \text{if } \alpha = \frac{1}{2} \end{cases}$$

In this chapter we have proved similar result for the classes $S(m, M)$ and $K(m, M)$ defined in Chapter 2. In proving our results we follow procedures developed by MacGregor. Actually our results contains the conjecture of Jack [22] .

3.2. We need the following lemmas for the proof of our theorem.

Lemma 3.2.1 :- Suppose that the functions T and S are analytic in D, $T(0) = 0 = S(0)$ and S maps D onto a (possibly many sheeted) region which is starlike with respect to the origin. If

$$(3.6) \qquad \operatorname{Re}\left\{\frac{T'(z)}{S'(z)}\right\} > \delta \qquad \text{for } |z| < 1$$

then

(3.7) $$\text{Re}\,\left\{\frac{T(z)}{S(z)}\right\} > \delta \quad \text{for } |z| < 1$$

and if

(3.8) $$\text{Re}\left\{\frac{T'(z)}{S'(z)}\right\} < \delta \quad \text{for } |z| < 1$$

then

(3.9) $$\text{Re}\left\{\frac{T(z)}{S(z)}\right\} < \delta \quad \text{for } |z| < 1$$

The first half of the lemma can be found in [30] and for $\delta = 0$ in [6] while completely it is in [36] .

Lemma 3.2.2 If we define $G(z)$ as following

(3.10) $$G(z) = \frac{az(1-bz)^{-\left(\frac{a+b}{b}\right)}}{(1-bz)^{-a/b} - 1}$$

then $G(z)$ is univalent.

Proof: If we write

(3.11) $$F(z) = \frac{(1-bz)^{-\frac{a}{b}} - 1}{a}$$

then, by logarithmic differentiation we get

(3.12) $$z\,\frac{F'(z)}{F(z)} = \frac{-\frac{a}{b}(1-bz)^{-\frac{a}{b}-1}(-b)z}{(1-bz)^{-a/b} - 1}$$

$$= \frac{az(1-bz)^{-\left(\frac{a+b}{b}\right)}}{(1-bz)^{-a/b} - 1} .$$

From (3.10) and (3.12) we have

$$(3.13) \qquad G(z) = \frac{z\, F'(z)}{F(z)}$$

Let us rewrite $G(z)$ in the following form

$$(3.14) \qquad G(z) = \frac{az}{(1-bz) - (1-bz)^{\frac{a+b}{b}}}$$

$$= \frac{-\left(\frac{a}{b}\right)}{1 + \frac{(1-bz)^{\left(\frac{a+b}{b}\right)} - 1}{bz}}$$

$$= \frac{-\left(\frac{a}{b}\right)}{1 + G_1(z)}$$

where

$$(3.15) \qquad G_1(z) = \frac{(1-bz)^{\left(\frac{a+b}{b}\right)} - 1}{bz}$$

Rewriting $G_1(z)$ in terms of $G_2(z)$ such that

$$(3.16) \qquad G_2(z) = G_1(z) + \frac{a+b}{b}$$

$$= \frac{a+b}{bz} \int_0^z \{1 - (1-bt)^{\left(\frac{a+b}{b}\right) - 1}\} dt.$$

$$= \frac{a(a+b)}{bz} \int_0^z G_3(t)\, dt$$

where

$$(3.17) \qquad G_3(z) = \frac{1-(1-bz)^{\frac{a+b}{b}-1}}{a}$$

Differentiating $G_3(z)$ and then differentiating logarith - mically and taking real part of both sides, we have

$$(3.18) \qquad \mathrm{Re}\{1+\frac{z\,G_3''(z)}{G_3'(z)}\} = \mathrm{Re}\{\frac{1-az}{1-bz}\}$$

$$= \frac{1-\mathrm{Re}(a+b)z+ab|z|^2}{|1-bz|^2}$$

$$\geq \frac{1-(a+b)\ |z|+ab|z|^2}{|1-bz|^2}$$

$$= \frac{(1-a|z|)\ (1-b|z|)}{|1-bz|^2} > 0$$

Since $G_3(0)=0$ and $G_3'(0)=1$ so $G_3(z)$ is univalent and convex. Using theorem 1.2.1 we observe that $G_2(z)$ is convex univalent (of-course not normalized). This in turn implies that $G(z)$ is univalent. As a remark we point out here that univalence of $G_2(z)$ and hence of $G(z)$ can also be established in the following way. As in (3.18), we find that

$$(3.19) \qquad \left|1+\frac{z\,G_3''(z)}{G_3'(z)}-\frac{1-ab}{1-b^2}\right| < \begin{cases} \dfrac{a-b}{1-b^2} & \text{if } a>b \\[2ex] \dfrac{b-a}{1-b^2} & \text{if } a<b \end{cases}$$

This implies that

$$(3.20) \qquad G_3(z) \in K\left(\frac{1-ab}{1-b^2}, \frac{|a-b|}{1-b^2} \right)$$

Then by theorem 2.3.1 it follows that

$$(3.21) \qquad G_4(z) \in K\left(\frac{1-ab}{1-b^2}, \frac{|a-b|}{1-b^2} \right)$$

where

$$(3.22) \qquad G_4(z) = \frac{2}{z} \int_0^z G_3(t)\, dt.$$

Since $G_2(z) = \dfrac{(a+b)\, G_4(z)}{2b}$, therefore $G_2(z)$ is univalent and hence $G(z)$ is univalent. But (3.21) is stronger conclusion than (3.18).

Lemma .2.3 : <u>If</u> $F(z)$ <u>is defined by</u> (3.11) **and** $a \leq b$ **then**

$$(3.23) \qquad \frac{1-(1+br)^{-\frac{a}{b}}}{ar} \leq \mathrm{Re}\left\{\frac{F(z)}{z}\right\} \leq \frac{(1-br)^{-\frac{a}{b}}-1}{ar}$$

<u>Proof</u> : Let $z = r e^{i\theta}$ and $1 - bz = Re^{i\phi}$. Then

$$(3.24) \qquad \mathrm{Re}\left\{\frac{F(z)}{z}\right\} = \frac{R^{-\frac{a}{b}}\, r\cos\left(\frac{a}{b}\phi + \theta\right) - r\cos\theta}{a\, r^2}$$

$$\equiv \quad P(r,\theta)$$

where $R = (1 - 2b\, r \cos\theta + b^2 r^2)^{\frac{1}{2}}$ and $\tan\phi = -\dfrac{br \sin\theta}{1 - br\cos\theta}$.

Therefore

$$\frac{dR}{d\theta} = \frac{br \sin\theta}{R} \tag{3.25}$$

and

$$\frac{d\phi}{d\theta} = \frac{br(br - \cos\theta)}{R^2} \quad . \tag{3.26}$$

Since $(1 - bz) = Re^{i\phi}$ and $Re(1 - bz) > 0$ so $\phi = 0$ if $\theta = 0$ or $\theta = \pi$.

From (3.24), we have

$$\frac{\partial P(r,\theta)}{\partial\theta} = \frac{-\frac{a}{b} R^{-(\frac{a+b}{b})} \frac{dR}{d\theta} \cos(\frac{a}{b}\phi+\theta) - R^{-\frac{a}{b}} \sin(\frac{a}{b}\phi+\theta) . (\frac{a}{b}\frac{d\phi}{d\theta} + 1) + \sin\theta}{a\, r^2} \tag{3.27}$$

$$= \frac{-\frac{a}{b} R^{-\frac{a+2b}{b}} . br \sin\theta \cos(\frac{a}{b}\phi+\theta) - R^{-\frac{a}{b}} \sin(\frac{a}{b}\phi+\theta) . (\frac{ar(br - \cos\theta)}{R^2} + 1) + \sin\theta}{a\, r^2}$$

$$= \frac{-ar\, R^{-\frac{a+2b}{b}} \cos(\frac{a}{b}\phi+\theta) - R^{-\frac{a}{b}} \sin(\frac{a}{b}\phi+\theta) . (\frac{ar(br - \cos\theta)}{R^2} + 1) + \sin\theta}{a\, r^2}$$

$$(3.28)\quad \left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=0} = \frac{-\,ar(1-br)^{-\left(\frac{a+2b}{b}\right)} - (1-br)^{-\left(\frac{a+2b}{b}\right)}(1-ar-br)^2+1}{a\,r}$$

$$= \frac{1-(1-br)^{-\frac{a+2b}{b}}\{ar+(1-ar-br)^2\}}{a\,r}$$

$$\equiv \frac{G_1(r)}{ar}$$

Differentiating $G_1(r)$ we get

$$(3.29)\quad G_1'(r) = -(1-br)^{-\left(\frac{a+2b}{b}\right)}\{a+2(1-ar-br)(-a-b)\}$$

$$-\ \frac{a+2b}{b}\cdot(1-br)^{-\frac{a+3b}{b}}\ b\{ar+(1-ar-br)^2\}$$

$$= -\,a(a+b)\,r\,(1-br)^{-\left(\frac{a+3b}{2}\right)}\{1+(a+b)r\}$$

If $a > 0$ then $G_1'(r) < 0$ and hence $G_1(r)$ is a decreasing function of r so $G_1(r) \leq G_1(0) = 0$. In this case $\left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=0} \leq 0$. Similarly, if $a < 0$ then $G_1'(r) > 0$ and hence $G_1(r)$ is an increasing function of r, so $G_1(r) \geq G_1(0) = 0$. In this case $\left.\frac{\partial^2 P(r,0)}{\partial\theta^2}\right|_{\theta=0} \leq 0$. This implies that $\mathrm{Re}\left\{\frac{F(z)}{z}\right\}$ is maximum if $\theta = 0$.

Now

$$(3.30)\quad \left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=\pi} = \frac{-\,ar(1+br)^{-\left(\frac{a+2b}{b}\right)} + (1+br)^{-\left(\frac{a+2b}{b}\right)}(1+ar+br)^2-1}{ar}$$

$$= - \frac{1-(1+br)^{-(\frac{a+2b}{b})}\{(1+ar+br)^2 - ar\}}{ar}$$

$$\equiv - \frac{G_2(r)}{ar}$$

Differentiating $G_2(r)$ with respect to r we have

$$(3.31) \quad G_2'(r) = -(1+br)^{-(\frac{a+2b}{b})}\{2(a+b)(1+ar+br) - a\}$$
$$+(a+2b)(1+br)^{-(\frac{a+3b}{b})}\{(1+ar+br)^2 - ar\}$$
$$= -a(a+b)r(1+br)^{-(\frac{a+3b}{b})}\{1-(a+b)r\}$$

Now two cases arise.

Case 1. $a+b<1$. In this case $\{1-(a+b)r\}>0$, so $G_2'(r)<0$ if $a>0$ and $G_2'(r)>0$ if $a<0$. If $a>0$, $G_2(r)$ is a decreasing function of r so $G_2(r) \leq G_2(0) = 0$ and hence $\left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=\pi} > 0$. If $a<0$, $G_2(r)$ is an increasing function of r so $G_2(r) \geq G_2(0) = 0$ and hence $\left.\frac{\partial^2 P(r,\theta)}{\partial\theta^2}\right|_{\theta=\pi} > 0$. This implies that $\mathrm{Re}\,\{\frac{F(z)}{z}\}$ is

minimum if $\theta = \pi$.

Case 2. $a+b>1$. In this case $0 < \frac{1}{a+b} < 1$ and a and b are positive. So we have $G_2(r) \leq \max.\{G_2(0), G_2(1)\}$

Now ,

$$G_2(1) \equiv Q(a,b) = 1 - (1+b)^{-\frac{a+2b}{b}} \{(1+a+b)^2 - a\}$$

$$\frac{\partial Q(a,b)}{\partial a} = (1+b)^{-\frac{a+2b}{b}} \left[\frac{\ln(1+b)}{b} - (1+2a+2b) \right]$$

$$\leq (1+b)^{-\frac{a+2b}{b}} \left[\frac{\ln(1+b)}{b} - (1+2b) \right]$$

$$= \frac{(1+b)^{-\frac{a+2b}{b}}}{b} \left[\ln(1+b) - b(1+2b) \right]$$

$$\equiv \frac{(1+b)^{-\frac{a+2b}{b}}}{b} \cdot H(b)$$

$$H'(b) = \frac{1}{1+b} - (1+4b) \leq 0.$$

Therefore $H(b)$ is a decreasing function of b so $H(b) \leq H(0) = 0$. Hence $\frac{\partial Q(a,b)}{\partial a} \leq 0$, so $Q(a,b)$ is a decreasing function of a so $Q(a,b) \leq Q(0,b) = 0$. Again since $G_2(0) = 0$ therefore $G_2(r) \leq 0$ hence $\left.\frac{\partial^2 P(r,\theta)}{\partial \theta^2}\right|_{\theta=\pi} > 0$ and hence the minimum of $\operatorname{Re}\{\frac{F(z)}{z}\}$ is attained if $\theta = \pi$. This completes the proof of the lemma.

<u>Lemma 3.2.4</u>. <u>If</u> $H(z) = \frac{1+az}{1-bz}$, $a \leq b$ <u>and</u> $G(z)$ <u>be defined by</u> (3.10) <u>then</u>

$$(3.32) \qquad H_k(z) = k\,H(z) + (1-k)\,G(z)$$

<u>is univalent in D for</u> $k \geq 1$.

<u>Proof</u> : We begin by showing

$$(3.33)\qquad \operatorname{Re}\left\{\frac{G'(z)}{H'(z)}\right\} < 1 \quad \text{for} \quad z \in D.$$

We see that

$$(3.34)\qquad G'(z) = \frac{a(1-bz)^{-\left(\frac{a+2b}{b}\right)}\{(1-bz)^{-\frac{a}{b}} - (1+az)\}}{\{(1-bz)^{-a/b} - 1\}^2}$$

and

$$(3.35)\qquad H'(z) = \frac{a+b}{(1-bz)^2}.$$

From (3.34) and (3.35) we get

$$(3.36)\qquad \frac{G'(z)}{H'(z)} = -\frac{1}{a+b}\cdot\frac{T(z)}{S(z)}$$

where

$$(3.37)\qquad T(z) = a(1-bz)^{-a/b}\{1 + az - (1-bz)^{-a/b}\}$$

and

$$(3.38)\qquad S(z) = S_1^2(z)$$

with

$$(3.39)\qquad S_1(z) = (1-bz)^{-a/b} - 1.$$

It is easy to see that $S_1(z) \varepsilon K(m, M)$ and hence belongs to $S(m, M)$ so $S(z)$ is twovalent and satisfies the condition

$$(3.40) \qquad \left| z \frac{S'(z)}{S(z)} - 2m \right| < 2M .$$

Now we compute $T'(z)/S'(z)$ as following :

$$(3.41) \qquad \frac{T'(z)}{S'(z)} = \frac{a(1-bz)\{1-(1-bz)^{-\frac{a+b}{b}}\} + a\{1+az-(1-bz)^{-\frac{a}{b}}\}}{2\{(1-bz)^{-a/b}-1\}}$$

$$= \frac{a[(a-b)z - 2\{(1-bz)^{-a/b}-1\}]}{2\{(1-bz)^{-a/b}-1\}}$$

$$= \frac{a(a-b)z}{2\{(1-bz)^{-a/b}-1\}} - a$$

$$= \frac{(a-b)z}{2F(z)} - a$$

where $F(z)$ is given by (3.11).

Now by using the result (3.24) in lemma 2.2.3, we get

$$(3.42) \qquad \mathrm{Re}\left\{\frac{T'(z)}{S'(z)}\right\} \geq \frac{a-b}{2} \cdot \frac{a}{1-(1+b)^{-a/b}} - a \quad \text{if } a \leq b.$$

From lemma 3.2.1 and (3.42) we get

$$(3.43) \qquad \mathrm{Re}\left\{\frac{T(z)}{S(z)}\right\} \geq \frac{a-b}{2} \cdot \frac{a}{1-(1+b)^{-a/b}} - a \qquad \text{if } a \leq b.$$

From (3.36) and (3.43) we have

$$(3.44) \qquad \mathrm{Re}\left\{\frac{G'(z)}{H'(z)}\right\} \leq -\frac{1}{a+b}\left\{\frac{\varepsilon-b}{2}\cdot\frac{a}{1-(1+b)^{-a/b}} - a\right\} \quad \text{if } a \leq b.$$

$$= \frac{a}{a+b} - \frac{a(a-b)}{2(a+b)\{1-(1+b)^{-a/b}\}}$$

To prove (3.33) it is sufficient to prove that

$$\frac{a}{a+b} - \frac{a(a-b)}{2(a+b)\{1-(1+b)^{-a/b}\}} \leq 1$$

or

$$(3.45) \qquad -\frac{a(a-b)}{2\{1-(1+b)^{-a/b}\}} \leq b$$

Since we are considering the case $b \geq a$ and we know that $(a+b) \geq 0$, so b is always positive and a may be positive and negative both. If $a \geq 0$, $\{1-(1+b)^{-a/b}\} \geq 0$ and if $a < 0$ then $\{1-(1+b)^{-a/b}\} < 0$ therefore (3.45) is equivalent to

$$(3.46) \qquad a^2 - ab + 2b - 2b(1+b)^{-a/b} > 0 \qquad \text{if } 0 < a < b$$

and

$$(3.47) \qquad a^2 - ab + 2b - 2b(1+b)^{-a/b} < 0 \qquad \text{if } a < b \text{ and } a < 0.$$

Let us write

$$(3.48) \qquad A(a,b) = a^2 - ab + 2b - 2b(1+b)^{-a/b}$$

Differentiating $A(a,b)$ with respect to a, we have,

$$(3.49)\qquad \frac{\partial A(a,b)}{\partial a} = 2a - b + 2(1+b)^{-a/b}\log(1+b)$$

and

$$(3.50)\qquad \frac{\partial^2 A(a,b)}{\partial a^2} = \frac{2}{b}\{b - (1+b)^{-a/b}\log^2(1+b)\}$$

$$\geq \frac{2}{b}\{b - \log^2(1+b)\} \qquad \text{if } a > 0$$

$$\equiv \frac{2}{b}U(b)$$

Also we have

$$(3.51)\qquad U'(b) = 1 - \frac{2\log(1+b)}{1+b}$$

$$= \frac{(1+b) - 2\log(1+b)}{1+b}$$

$$\equiv \frac{V(b)}{1+b}$$

and

$$(3.52)\qquad V'(b) = 1 - \frac{2}{1+b}$$

$$= -\frac{1-b}{1+b} < 0$$

Thus it follows that $V(b)$ is a decreasing function of b and hence

$$V(b) \geq V(1) = 2(1-\log 2) > 0.$$

Thus $U'(b) > 0$. Hence $U(b)$ is an increasing function of b. But $\min_b U(b) = U(0) = 0$ hence $\frac{\partial A(a,b)}{\partial a}$ is an increasing function of a for all fixed b. But

$$(3.53) \qquad \frac{\partial A(a,b)}{\partial a} \geq \left[\frac{\partial A(a,b)}{\partial a}\right]_{a=0} = -b + 2 \log(1+b)$$

$$\equiv T(b)$$

Clearly $T'(b) = \frac{1-b}{1+b} > 0$ and so $T(b) \geq T(0) = 0$. Thus $\frac{\partial A(a,b)}{\partial a} > 0$. Hence $A(a,b)$ is an increasing function of a. So $A(a, b) \geq A(0, b) = 0$. Thus (3.46) is proved.

Situation in case $a < 0$, $a \leq b$ is slightly different. Since neither $\frac{\partial A(a,b)}{\partial a}$ nor $\frac{\partial A(a,b)}{\partial b}$ are purely increasing or decreasing function, we shall determine the sign of thesecond derivative.

From (3.50) we have

$$(3.54) \qquad \frac{\partial^2 A(a, b)}{\partial^2 a} \geq \frac{2}{b}\{b - (1+b) \log^2(1+b)\}$$

$$\equiv B(b) .$$

Now,

$$B'(b) = \frac{2}{b^2} \log^2(1+b) - \frac{2}{b} \cdot 2 \log(1+b)$$

$$= -\frac{2}{b^2} \log(1+b) \{2b - \log(1+b)\}$$

$$= -\frac{2}{b^2} \log(1+b)\, U_1(b) .$$

Also $U_1'(b) = 2 - \frac{1}{1+b} = \frac{1+2b}{1+b} > 0$. This implies that $U_1(b)$ is an increasing function of b so $U_1(b) \geq U_1(0) = 0$. Hence $B'(b) < 0$.

Thus $B(b) \geq B(1) = 2(1-2\log^2 2) \geq 0$. Therefore the second derivative of $A(a,b)$ is positive. Now $a+b \geq 0$ and we are considering $a \leq b$, $a < 0$ so $0 \geq a \geq -b$ and $A(0,b) = 0 = A(-b, b)$. Hence, by Roll's theorem, it follows that $A(a,b)$ is positive in $-b < a < 0$. This completes the proof of the fact

$$\mathrm{Re}\left\{\frac{G'(z)}{H'(z)}\right\} < 1 \quad \text{in } D.$$

Now we show that H_k is univalent. Clearly $H(z) = 1+(a+b)N(z)$ and $N(z)$ is convex, gives H is convex in D. Since H is convex and (3.37) is satisfied in the case $a \leq b$, it follows from the argument of Pommerenke [51]

$$\mathrm{Re}\left\{\frac{G(z_2) - G(z_1)}{H(z_2) - H(z_1)}\right\} < 1 \quad \text{for } z_1, z_2 \in D \tag{3.55}$$

Let us assume $H_k(z)$ is not univalent. Then, we must have for $z_1 \neq z_2$, $H_k(z_1) = H_k(z_2)$ for some z_1 and z_2 in D. This implies that

$$\frac{G(z_2) - G(z_1)}{H(z_2) - H(z_1)} = \frac{k}{k-1} > 1 \tag{3.56}$$

But (3.56) contradicts (3.55). Hence, $H_k(z)$ must be univalent. This completes the proof of the lemma.

<u>Lemma 3.2.5</u> : $H(z)$ <u>is</u> $\ll H_k(z)$ <u>in</u> D.

Proof : Since H_k is univalent by lemma 3.2.4 and $H(0) = H_k(0)$, the subordination follows if $H(D) \subset H_k(D)$. Clearly, H maps D onto the circle $|w-m| < M$. Also if $z = e^{i\theta}$, then, we obtain

$$(3.57) \qquad |w - m| = \left| \frac{1 + a\, e^{i\theta}}{1 - b\, e^{i\theta}} - m \right|$$

$$= \left| \frac{(1-m) + (a \pm bm)\, e^{i\theta}}{1 - b\, e^{i\theta}} \right|$$

$$= M \left| \frac{(1-m) + M\, e^{i\theta}}{M-(m-1)\, e^{i\theta}} \right|$$

$$= M.$$

Hence, H maps the boundary of D onto the boundary of the circle $|w-m| < M$. Thus, the lemma will be proved if we show that the points in boundary of H_k satisfy

$$(3.58) \qquad |w - m| \geq M.$$

Suppose $|z_1| = 1$, $w_1 = \lim_{z \to z_1} H(z)$, $w_2 = \lim_{z \to z_2} G(z)$. Now we want to prove

$$(3.59) \qquad |k\, w_1 + (1-k) w_2 - m| \geq M$$

or

$$|k(w_1 - m) + (1-k)(w_2 - m)| \geq M$$

(3 59) will be satisfied if

$$(3.60) \qquad |k|\,|w_1 - m| - |1-k|\,|w_2 - m| \geq M .$$

Using (3.57) in (3.60), we see that (3.59) is satisfied if

$$kM - |1-k||w_2-m| \geq M.$$

Thus inequality follows if $|w_2 - m| \leq M$. However, this is obviously true from theorem 1.2.1. Hence the lemma is proved.

3.3 In this section we shall prove the following theorem.

Theorem 3.3.1 : If $f \in K(m, M)$ and G is defined by (3.10) then $z\frac{f'(z)}{f(z)} \prec\prec G(z)$ in D for $b \geq a$.

Proof : We shall follow similar lines of proof as developed by T.H. MacGregor [36] . If we write $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where $F(z)$ is defined by (3.11) and $f \in K(m, M)$, then $A_2 = a + b$ and is well known $|a_2| \leq A_2$. It is obvious that $|a_2| = A_2$, if and only if,

$$(3.61) \qquad f(z) = \frac{1}{a} e^{-i\eta} \{(1-b\, e^{i\eta} z)^{-a/b} - 1\}, \qquad \eta \text{ is real.}$$

This result is due to Z.J. Jakubowski [20] . Now if $g(z) = \frac{zf'(z)}{f(z)}$ and $G(z)$ is defined by (3.10) , and further, if we write

$$(3.62) \qquad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

and

$$(3.63) \qquad G(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$$

then $|b_1| \leq B_1$ is equivalent to $|a_2| \leq A_2$. Since $b_1 = a_2$ and $B_1 = A_2$, if follows that $|b_1| < B_1$ is equivalent to $|a_2| < A_2$. Also $|b_1| = B_1$ only it $g(z) = G(e^{i\eta}z)$ where η is real . As the function

$$(3.64) \qquad G(e^{i\eta}z) << G(z)$$

η real, we continue the argument by assuming $|b_1| < B_1$.

If we set $\Delta r = \{z : |z| < r\}$ (obviously $\Delta_1 = D$) then $|b_1| < B_1$ implies that $g(\Delta_r) \subset G(\Delta_r)$ for sufficiently small valuesof r. This subordination implies that there exists a $w(z)$ such that

$$(3.65) \qquad w(z) = G^{-1}(g(z))$$

and $w(z)$ is analytic for $|z| < r$ and satisfies $w(0) = 0$ and

$$(3.66) \qquad |w(z)| < r$$

for sufficiently small values of r.

Let $\rho = \text{Sup. } r$ where $0 < r < 1$ and w be analytic for $|z| \leq r$ and satisfy (3.66) for $|z| < r$. We need only to show that $\rho = 1$. On contrary let us assume $0 < \rho < 1$. Then $w(z)$ is analytic for $|z| < \rho$ and $|w(z)| < \rho$ for $|z| < \rho$. We first show that $w(z)$ is analytic for $|z| \leq \rho$. We know that $g << G$ in Δ_ρ and thus

$$(3.67) \qquad g(\bar{\Delta}_\rho) \subset G(\bar{\Delta}_\rho)$$

Since $G(\overline{\Delta}_\rho) \subset G(\Delta_1)$ it follows that $g(\Delta_{\rho+\varepsilon}) \subset G(\Delta_1)$ for all sufficiently small values of $\varepsilon (\varepsilon > 0)$. Therefore because G is univalent in Δ_1, equation (3.65) defines w as an analytic function on $\Delta_{\rho+\varepsilon}$. Since w is analytic for $|z| \leq \rho$, the definition of ρ implies that there is a number z_1 such that $|z_1| = \rho$ and $|w(z_1)| = \rho$. Then by Jack's lemma 2.2.1 there exists a real number k such that

$$(3.68) \qquad z_1 w'(z_1) = k\, w(z_1) \quad \text{for some } z_1 \text{ and } k \geq 1.$$

Since $h \prec\prec H$, where H is defined by $H(z) = \frac{1+az}{1-bz}$ and

$$(3.69) \qquad h(z) = 1 + z\frac{f''(z)}{f'(z)}$$

in D. We may write $h(z) = H(\phi(z))$ where $\phi(z)$ is analytic in D, $|\phi(z)| < 1$ and $\phi(0) = 0$. Writing in terms of g, we may express $h(z) = H(\phi(z))$ in the form

$$(3.70) \qquad z\frac{g'(z)}{g(z)} + g(z) = H(\phi(z)).$$

Equation (3.65) implies that $g(z) = G(w(z))$ and $g'(z) = G'(w(z))\, w'(z)$. If we use these relations at $z = z_1$, then, we have

$$(3.71) \qquad \frac{k\, w(z_1)\, G'(w(z_1))}{G(w(z_1))} + G(w(z_1)) = H(\phi(z_1)).$$

Since $H(z) = z\frac{G'(z)}{G(z)} + G(z)$, equation (3.71) is the same as

$$(3.72) \qquad H_k(w(z_1)) = H(\phi(z_1))$$

where $H_k(z)$ is defined in (3.36). Because of lemma 3.2.4, $\psi = H_k^{-1}(H(\phi))$ is analytic in D, $|\psi(z)| < 1$ and $\psi(0) = 0$ equation (3.72) implies that

$$(3.73) \qquad H_k(w(z_1)) = H_k(\psi(z_1))$$

and since H_k is univalent in D and $w(z_1)$ and $\psi(z_1)$ are equal. In particular it follows that $|\psi(z_1)| = |w(z_1)| = \rho = |z_1|$. Equality in Schwarz's lemma is possible only if $\psi(z) = z\, e^{i\delta}$, δ real. Thus we have

$$\begin{aligned} H_k(\psi) &= k\, H(\psi) + (1-k)\, G(\psi) \\ &= k\, H(z\, e^{i\delta}) + (1-k)\, G(z\, e^{i\delta}) \end{aligned}$$

Also

$$\begin{aligned} H(e^{i\delta} z) &= \frac{1 + a\, z\, e^{i\delta}}{1 - b\, z\, e^{i\delta}} \\ &= 1 + (a+b)\, z\, e^{i\delta} + \ldots\ldots \end{aligned}$$

and

$$G(e^{i\delta} z) = 1 + (b-1)\, z\, e^{i\delta} + \ldots\ldots$$

hence it follows that

$$H_k(\psi) = 1 + \{ k(a+b) + (1-k)\,(b-1) \}\, z\, e^{i\delta} + \ldots\ldots$$

Now, if $\phi(z) = c_1 z + c_2 z^2 + \ldots$ then by comparing coefficients in $\psi = H_k^{-1}(H(\phi))$ we obtain

$$(a+b)c_1 = [(k-1) + (1-a)k]\, e^{i\delta}$$

This equation gives

$$k = \frac{(a+b)c_1 e^{-i\delta} + (1-b)}{1+a}$$

But $k \geq 1$, therefore, we must have

$$c_1 e^{-i\delta} \geq 1.$$

But for bounded function $\Phi(z)$ we know $|c_1| \leq 1$. Hence

$$|c_1| = 1 \quad \text{or} \quad c = e^{i\delta}$$

Hence $h(z) = H(e^{i\delta} z)$. This yields for all real δ, $|b_i| = B_1$. This is a contradiction. Hence, we must have $\rho = 1$. This proves the theorem.

If $f(z)$ is in $K(m, M)$ then from theorem 3.3.1 we have $f(D) \subset G(D)$ hence we get following results as corollaries.

<u>Corollary 3.3.1</u> : <u>If</u> $f(z)$ <u>belongs to</u> $K(m, M)$ <u>and</u> $b \geq a$ <u>then</u>

$$\frac{a}{(1+br)\{(1+br)^{a/b} - 1\}} \leq \left|\frac{f'(z)}{f(z)}\right| \leq \frac{a}{(1-br)\{1-(1-br)^{a/b}\}}$$

<u>Corollary 3.3.2</u> : <u>If</u> f(z) <u>belongs to</u> K(m, M) <u>and</u> $b \geq a$ <u>then</u> f(z) <u>belongs to</u> S(m', M') <u>where</u>

$$m' = \frac{1}{2}\left[\frac{a}{(1-b)\{1-(1-b)^{a/b}\}} + \frac{a}{(1+b)\{(1+b)^{a/b}-1\}}\right]$$

<u>and</u>

$$M' = \frac{1}{2}\left[\frac{a}{(1-b)\{1-(1-b)^{a/b}\}} - \frac{a}{(1+b)\{(1+b)^{a/b}-1\}}\right].$$

CHAPTER - 4

ON RADIUS OF STARLIKENESS OF SOME CLASSES OF FUNCTIONS.

4.1 In this chapter we shall prove weak converse of theorems 2.3.1 and theorem 2.3.3 and converse of theorem 2.3.2. In the proof of a theorem of this chapter we need the following lemma.

Lemma 4.1.1. *If* $g(z) \in K(\alpha)$ *then*

$$(4.1)\qquad \left|\frac{zg'(z)}{g(z)}\right| \le B(\alpha,r) = \begin{cases} \dfrac{(2\alpha-1)r}{(1-r)^{2(1-\alpha)}\{1-(1-|z|)^{2\alpha-1}\}}, & \alpha \ne \dfrac{1}{2} \\[2ex] -\dfrac{r}{(1-r)\log(1-r)}, & \alpha = \dfrac{1}{2}. \end{cases}$$

Proof : We have stated a result of T.H. Mac Gregor in chapter 3 as theorem D which gives that $\frac{zg'(x)}{g(z} \ll J(z)$ where $J(z)$ is given by (3.5). Hence

$$(4.2)\qquad \left|\frac{zg'(z)}{g(z)}\right| \le |J(z)| \le B(\alpha,r).$$

4.2. In this section we prove the following theorems.

Theorem 4.2.1. *Let* $F \in S(m,M)$ *and* $f(z)$ *be defined by* (2.18) *and* $r(a,b)$ *be the unique positive root of the equation*

$$(4.3)\qquad (a+2b+d)-2(ad+bd+b+d)r - \{2(b^2-d^2) + (a+d) + 2b(1-d^2)$$

$$- d(ad+b^2)\} r^2 - 2d \{(a+b) + b(b+d) \quad r^3\}$$

$$- d(ad + 2bd + b^2)\, r^4 = 0$$

then, $f(z)$ is starlike of order β for $|z| < r_o$, where r_o is the smallest positive root of the equation

$$(4.4) \qquad (1-\beta) - \{\beta(b-d) + a+b-2d\}r + d(a+b\beta)\,r^2 = 0$$

if $r_o \leq r(a,b)$, otherwise r_o is the smallest positive root of the equation

$$(4.5) \qquad (E - 1+bd) - (1+bd)x + \sqrt{(1-d)\,\{(1-d) + (1+d)x\}\,\{(1+2a+4b-b^2)+(1+b^2)x\}} = 0$$

where

$$x = \frac{1 + r^2}{1 - r^2}, \quad E = -\beta(b+d) + 2d - (a+b) \text{ and } d = \frac{a - bc}{c+1}.$$

This result is sharp.

Proof : Since $F \in S(m,M)$ there exists a regular function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ and

$$(4.6) \qquad \frac{zF'(z)}{F(z)} = \frac{1 + a\,w(z)}{1 - b\,w(z)}$$

From (2.18) we have

$$z^c\,F'(z) + c\,z^{c-1}\,F(z) = (c+1)\,z^{c-1}\,f(z)$$

or

$$(4.7) \qquad (c+1)\,\frac{f(z)}{F(z)} = c + \frac{zF'(z)}{F(z)}.$$

From (4.6) and (4.7) we get

$$(4.8) \qquad \frac{f(z)}{F(z)} = \frac{1 + \frac{a-bc}{c+1} w(z)}{1 - bw(z)}$$

$$= \frac{1 + dw(z)}{1 - bw(z)}$$

Differentiating (4.8) logarithmically with respect to z and using (4.6), we get,

$$\frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)} + \frac{(b+d)zw'(z)}{(1-bw(z))(1+dw(z))}$$

or

$$\frac{zf'(z)}{f(z)} - \beta = -\beta + \frac{1 + aw(z)}{1 - bw(z)} + \frac{(b+d)w(z)}{(1-bw(z))(1+dw(z))}$$

$$+ \frac{(b+d)(zw'(z) - w(z))}{(1-bw(z))(1 + dw(z))}$$

or

$$(4.9) \qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq -\beta + \mathrm{Re}\left\{\frac{1+aw(z)}{1-bw(z)}\right\}$$

$$+ (b+d)\,\mathrm{Re}\left\{\frac{w(z)}{(1-bw(z))(1+dw(z))}\right\} - \frac{(b+d)(r^2-|w(z)|^2)}{(1-r^2)|1-bw(z)|\,|1+dw(z)|}$$

Here we have used the well known inequality $|zw'(z) - w(z)| \leq \frac{r^2-|w(z)|^2}{1 - r^2}$ due to Singh and Goel [59].

If we take

$$(4.10) \qquad p(z) = \frac{1 + dw(z)}{1 - bw(z)}$$

it is easy to see that

$$(4.11) \qquad |p(z) - A| \leq B$$

where

$$(4.12) \qquad A = \frac{1 + bdr^2}{1 - b^2r^2}$$

and

$$(4.13) \qquad B = \frac{(b+d)r}{1-b^2r^2}.$$

Substituting value of $w(z)$ from (4.10) in (4.9) we get

$$(4.14) \qquad \mathrm{Re}\,\{\frac{zf'(z)}{f(z)} - \beta\} \geq \frac{1}{b+d}\,[E - d\,\mathrm{Re}\,\{\frac{1}{p(z)}\} + (a+2b)\,\mathrm{Re}\,\{p(z)\} - \frac{r^2\,|bp(z) + d|^2 - |p(z) - 1|^2}{(1-r^2)\,|p(z)|}].$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$ and use (4.12) and (4.13) in (4.14) we get

$$(4.15) \qquad \mathrm{Re}\,\{\frac{zf'(z)}{f(z)} - \beta\} \geq \frac{1}{b+d}\,[E - \frac{d(A+u)}{R^2} + (a+2b)(A+u) - \frac{B^2 - u^2 - v^2}{R}\,(\frac{1 - b^2r^2}{1 - r^2})] \equiv \frac{1}{b+d}\cdot P(u,v).$$

Differentiating $P(u,v)$ partially with respect to v we get

$$(4.16) \qquad \frac{\partial P(u,v)}{\partial v} = \frac{v}{R}\,[\frac{2d(A+u)}{R^3} + \{2 + \frac{B^2 - u^2 - v^2}{R}\}\,(\frac{1 - b^2r^2}{1 - r^2})]\,.$$

If $d \geq 0$, quantity in the square bracket is positive. If $d < 0$ we see that

$$\frac{1-b^2r^2}{1-r^2} + \frac{d(A+u)}{R^3} \geq 1 + \frac{d(1+br)^2}{(1-dr)^2}$$

$$= \frac{(1+d) - 2d(1-b)r + d(b^2+d)r^2}{(1-dr)^2}$$

$$= \frac{(1+d)\ \{1 + \frac{d(b^2+d)}{(1+d)}\ r^2\} - 2d(1-b)r}{(1-dr)^2}$$

$$\geq 0$$

and therefore the quantity in the square bracket in (4.16) is positive. So $\frac{\partial P(u,v)}{\partial v} \geq 0$ if $v \geq 0$ and $\frac{\partial P(u,v)}{\partial v} < 0$ if $v < 0$ therefore

(4.17) $$\min_v P(u,v) = P(u,0)$$

$$= E - \frac{d}{R} + (a+2b)R - \frac{B^2-(R-A)^2}{R}\left(\frac{1-b^2r^2}{1-r^2}\right)$$

$$\equiv P(R),$$

where $R = A + u$. Now

$$P'(R) = (a+2b) + \frac{d}{R^2} - \left(\frac{A^2-B^2}{R^2}\right)\left(\frac{1-b^2r^2}{1-r^2}\right) + \frac{1-b^2r^2}{1-r^2}$$

and

$$P''(R) = -\frac{2d}{R^3} + 2\frac{(A^2-B^2)}{R^3} \cdot \left(\frac{1-b^2r^2}{1-r^2}\right)$$

$$= \frac{2}{R^3}\ \frac{(1-d)(1+dr^2)}{1-r^2}$$

$$\geq 0\ .$$

where $x = \dfrac{1+r^2}{1-r^2}$.

Let us take

$$(4.20)\quad Q(r) = (A-B)^2 - R_o^2$$

$$= \left(\frac{1-dr}{1+br}\right)^2 - \frac{(1-d)(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}$$

$$= \frac{(a+2b+d)-2(ad+bd+b+d)r-\{2(b^2-d^2)+(a+d)+2b(1-d^2)-d(ad+b^2)\}r^2 + 2d\{(a+b)+b(b+d)\}\, r^3 - d(ad+2bd+b^2)r^4}{(1+br)^2\{(a+2b+1)-(a+2b+b^2)r^2\}}$$

$$Q'(r) = -2\left(\frac{1-dr}{1+br}\right)\frac{b+d}{(1+br)^2} - \frac{2(1-d)\{(a+b)(1+d)+(1+b)(b+d)\}\, r}{\{(a+2b+1)-(a+2b+b^2)r^2\}^2}$$

$$\leq 0 .$$

Therefore $Q(r)$ is a decreasing function of r and $Q(0) = \dfrac{(a+b)+(b+d)}{(a+b)+(1+b)} \geq 0$ and $Q(1) = -\dfrac{2(1-d)(b+d)}{(1+b)(1-b^2)} \leq 0$. Therefore $Q(r)$ has unique root in $(0,1)$.

Let it be $r(a,b)$. Hence if $r \leq r(a,b)$, $Q(r) \geq 0$ i.e. $A-B \geq R_o$ and if $r \geq r(a,b)$ $Q(r) \leq 0$ i.e. $A-B \leq R_o$. So from (4.19) and (4.20) the result follows.

The equality in (4.4) is attained for the function

$$F(z) = z(1-bz)^{-\frac{a+b}{b}}$$

and that in (4.5) for the function

$$F(z) = z(1 - 2k\ bz + b^2z^2)^{-\frac{a+b}{2b}}$$

where k is given by

$$\frac{1 + k(a-b)r - br^2}{1 - 2kbr + b^2 r^2} = \left\{\frac{(1-d)(1+dr^2}{(a+2b+1)-(a+2b+b^2)r^2}\right\}^{1/2} .$$

If we put $a = 2\alpha - 1$, $b = 1$ and $c = 1$ in this theorem the result of P.L. Bajpai and P. Singh [4] follows and if we put $a = 2\beta - 1$, $b = 1$ the result of S.K. Bajpai and R.S.L. Srivastava [3] follows.

Theorem 4.2.2. *If $f(z)$ is regular in D and satisfy (2.33) where $F \in S^*(\beta)$ and $g \in S(m,M)$ then $f(z)$ is univalent and starlike of order β in $|z| < r_o$ where r_o is the smallest positive root of the equation*

$$(4.21) \quad (1-\beta)(c+2)-\{(c+2)(a+2b-b\beta)+2(1-\beta)(2-\beta)\}r+\{2b(1-\beta)(2-\beta) - (1-\beta)(c+2\beta)-2(c+1+\beta)(a+b)\}r^2-(c+2\beta)(a+b\beta)r^3 = 0$$

This result is sharp.

Proof : From (2.33) we have

$$(4.22) \qquad (c+2)\frac{f(z)g(z)}{zF z)} = (c+1) + \frac{zF'(z)}{F(z)} .$$

Since $F \in S^*(\beta)$ there exists a regular function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ and

$$(4.23) \qquad \frac{zF' z)}{F z)} = \frac{1 + (2\beta-1) w z)}{1+w(z)} .$$

From (4.22) and (4.23) we get

$$(4.24) \qquad \frac{f(z) g(z)}{z F(z)} = \frac{1 + \frac{c + 2\beta}{c+2} w(z)}{1 + w z)} .$$

Differentiating (4.24) logarithmically with respect to z and using (4.23) we get

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zg'(z)}{g(z)} + \frac{1+(2\beta-1)w(z)}{1+w(z)} + \frac{c+2\beta}{c+2} \cdot \frac{zw'(z)}{1+\frac{c+2\beta}{c+2}w(z)} - \frac{zw'(z)}{1+w(z)}$$

or

$$\left\{\frac{zf'(z)}{f(z)} - \beta\right\} = 1+(1-\beta)\left[\frac{1-w(z)}{1+w(z)} - \frac{2}{c+2}\,\frac{zw'(z)}{\{1+w(z)\}\{1+\frac{c+2\beta}{c+2}w(z)\}}\right] - \frac{zg'(z)}{g(z)}$$

or

$$(4.25)\quad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq 1+(1-\beta)\left[\frac{1-|w(z)|^2}{|1+w(z)|^2} - \frac{2}{c+2}\left|\frac{zw'(z)}{\{1+w(z)\}\{1+\frac{c+2\beta}{c+2}w(z)\}}\right|\right] - \left|\frac{zg'(z)}{g(z)}\right|.$$

Using the well known inequality $|w'(z)| \leq \frac{1-|w(z)|^2}{1-r^2}$ [40 page 168] in (4.25) we get

$$(4.26)\quad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq 1 + \frac{(1-\beta)(1-|w(z)|^2)}{|1+w(z)|\,\left|1+\frac{c+2\beta}{c+2}w(z)\right|}\left[\left|\frac{1+\frac{c+2\beta}{c+2}w(z)}{1+w(z)}\right| - \frac{2r}{(c+2)(1-r^2)}\right] - \left|\frac{zg'(z)}{g(z)}\right|$$

It is easy to see that

$$(4.27)\quad \left|\frac{1+\frac{c+2\beta}{c+2}w(z)}{1+w(z)}\right| \geq \frac{1+\frac{c+2\beta}{c+2}r}{1+r}$$

and since $g \in S(m,M)$ we have

$$(4.28) \qquad \left|\frac{zg'(z)}{g(z)}\right| \leq Q(r) = \frac{1+ar}{1-br}.$$

From (4.26), (4.27) and (4.28) we have

$$(4.29) \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq 1-Q(r) + \frac{(1-\beta)(1-|w(z)|^2)}{|1+w(z)|\,\left|1+\frac{c+2\beta}{c+2}w(z)\right|}\left[\frac{1+\frac{c+2\beta}{c+2}r}{1+r} - \frac{2r}{(c+2)(1-r^2)}\right]$$

$$= 1-Q(r) + \frac{(1-\beta)(1-|w(z)|^2)\left\{1-\frac{2(2-\beta)}{c+2}r - \frac{c+2\beta}{c+2}r^2\right\}}{|1+w(z)|\,\left|1+\frac{c+2\beta}{c+2}w(z)\right|(1-r^2)}.$$

From (4.29) it follows that $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} > 0$

if

$$1-Q(r) \geq -\frac{(1-\beta)(1-|w(z)|^2)\left\{1-\frac{2(2-\beta)}{c+2}r - \frac{c+2\beta}{c+2}r^2\right\}}{|1+w(z)|\,\left|1+\frac{c+2\beta}{c+2}w(z)\right|(1-r^2)}$$

i.e. if

$$(4.30) \quad 1-Q(r) \geq -\frac{(1-\beta)\left\{1-\frac{2(2-\beta)}{c+2}r - \frac{c+2\beta}{c+2}r^2\right\}}{(1+r)\left\{1+\frac{c+2\beta}{c+2}r\right\}}$$

provided $r < r(\beta)$ where $r(\beta)$ is the positive root of the equation

$$1 - \frac{2(2-\beta)}{c+2}r - \frac{c+2\beta}{c+2}r^2 = 0.$$

Let us take

$$(4.31) \qquad P(r) = 1 - Q(r) + \frac{(1-\beta)\left\{1 - \frac{2(2-\beta)}{c+2} r - \frac{c+2\beta}{c+2} r^2\right\}}{(1+r)\left(1 + \frac{c+2\beta}{c+2} r\right)} .$$

Since $P(0) = 1 - Q(0) + (1-\beta)$

$$= (1-\beta) > 0$$

and

$$P(r(\beta)) = 1 - Q(r(\beta))$$

$$= 1 - \frac{1 + ar(\beta)}{1 - br(\beta)}$$

$$= -\frac{(a+b)\, r(\beta)}{1-br(\beta)} < 0 ,$$

therefore smallest positive root of the equation $P(r) = 0$ is less than $r(\beta)$. But

$$P(r) = \frac{(1-\beta)(c+2)-\{(c+2)(a+2b-b\beta)+2(1-\beta)(2-\beta)\}\, r+\{2b(1-\beta)(2-\beta) -(1-\beta)(c+2\beta)-2(c+1+\beta)(a+b)\}r^2 -(c+2\beta)(a+b\beta)r^3}{(1+r)(1-br)(c+2+(c+2\beta)r)}$$

So $P(r) > 0$ if $r < r_0 < r(\beta)$. This completes the proof of the theorem.

Equality in (4.21) is attained for the functions

$$F(z) = \frac{z}{(1-z)^{2(1-\beta)}}$$

and

$$g(z) = \frac{z}{(1+bz)^{\frac{a+b}{b}}}$$

From theorem 4.2.2 we obtain theorem 2 of Calys [11] as a corollary by taking $a = b = 1$, $c = 0$, $\beta = 0$.

Corollary 4.2.1. If $f(z)$ is regular in D and satisfies (2.33) where $F \in S^*(\beta)$ and $g \in S$ then $f(z)$ is univalent and starlike of order β in $|z| < r_o$ where r_o is the smallest positive root of the equation,

$$(1-\beta)(c+2)-\{(c+2)(3-\beta)+2(1-\beta)(2-\beta)\}r+\{2(1-\beta)(2-\beta)-(1-\beta)(c+2\beta)$$
$$-4(c+1+\beta)\}\, r^2 - (1+\beta)(c+2\beta)\, r^3 = 0 .$$

The result is sharp.

Proof : If $g \in S$ then we know that [19]

$$\left|\frac{zg'(z)}{g(z)}\right| \leq \frac{1+r}{1-r} .$$

We can get this bound from the bound of $g \in S(m,M)$ by taking $a = b = 1$. So if we substitute $a = b = 1$ in theorem 4.2.2 we get the result in this corollary.

Theorem 4.2.3. If $f(z)$ is regular in D and satisfies (2.33) where $F \in S^*(\beta)$ and $g \in K(\alpha)$, then $f(z)$ is starlike of order β for $|z| < r_o$, where r_o is the smallest positive root of the equation

$$(4.32)\quad (c+2)(2-\beta)+2\{(c+\beta+1)-(1-\beta)(2-\beta)\}r + \beta(c+2\beta)\, r^2 - (1+r)\,\{(c+2) + (c+2\beta)r\}\, B(\alpha,r) = 0$$

where

$$(4.33)\quad B(\alpha,r) = \begin{cases} \dfrac{(2\alpha-1)r}{(1-r)^2(1-\alpha)\{1-(1-r)^{2\alpha-1}\}}, & \alpha \neq \dfrac{1}{2} \\[2ex] \dfrac{r}{(1-r)\log(1-r)}, & \alpha = \dfrac{1}{2} \end{cases}$$

<u>This result is sharp.</u>

<u>Proof</u> : Since $g \in K(\alpha)$ from lemma 4.1.1 we have

$$(4.34) \qquad \left|\frac{zg'(z)}{g(z)}\right| \le B(\alpha,r)$$

where $B(\alpha,r)$ is given in (4.33). Proceeding on the same lines as in theorem 4.2.2 and using (4.34) we get

$$(4.35) \qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} > 0$$

if

$$(4.36) \qquad 1 - B(\alpha,r) \ge - \frac{(1-\beta)\left\{1 - \frac{2(2-\beta)}{c+2} r - \frac{c+2\beta}{c+2} r^2\right\}}{(1+r)\left(1 + \frac{c+2\beta}{c+2} r\right)}$$

and

$$(4.37) \qquad 1 - B(\alpha, r(\beta)) < 0$$

where $r(\beta)$ is the positive root of the equation

$$1 - \frac{2(2-\beta)}{c+2} r - \frac{c+2\beta}{c+2} r^2 = 0.$$

From (4.35) and (4.36) we get (4.32). So we get the result if (4.37) holds. To prove (4.37) it is sufficient to prove that $\underset{0 \le r < 1}{\text{g.l.b}}\ B(\alpha,r) \ge 1$

Now

$$\frac{d}{dr} B(\alpha,r) = \frac{B(\alpha,r)}{1-r}\left[1 - (2\alpha-1) r - \frac{(2\alpha-1) r}{\{(1-r)^{1-2\alpha} - 1\}}\right]$$

So,

$\frac{dB}{dr} > 0$ if

$$1 - (2\alpha-1)\, r > \frac{(2\alpha-1)r}{(1-r)^{-2\alpha} - 1}\ .$$

Case 1. $\alpha > \frac{1}{2}$, then $(1-r)^{1-2\alpha} - 1 > 0$, therefore

$$\frac{dB}{dr} > 0 \text{ if } (1-r)^{1-2\alpha} - 1 - (2\alpha-1)\, r(1-r)^{1-2\alpha} > 0$$

i.e. if $h(r) \equiv 1 - (1-r)^{2\alpha-1} - (2\alpha-1)r > 0.$

But

$$h'(r) = (2\alpha-1)\,[(1-r)^{-2(1-\alpha)} - 1] > 0$$

hence $h(r)$ is an increasing function of r. Therefore

$$h(r) \geq \min_{0 \leq r < 1} h(r) = h(0) = 0.$$

Case 2 : $\alpha < \frac{1}{2}$ then $(1-r)^{1-2\alpha} - 1 < 0$, therefore,

$$\frac{dB}{dr} > 0 \text{ if } h(r) = 1 - (1-r)^{2\alpha-1} - (2\alpha-1)r < 0.$$

But

$$h'(r) = (2\alpha-1)\,[(1-r)^{-2(1-\alpha)} - 1] < 0$$

hence $h(r)$ is a decreasing function of r therefore

$$h(r) < \max_{0 \leq r < 1} h(r) = h(0) = 0.$$

Hence $\frac{dB(\alpha,r)}{dr} > 0$ for every $\alpha \in [0,1)$ and $\alpha \neq \frac{1}{2}$. Therefore $B(\alpha,r)$ is an increasing function of r. Hence $\underset{0 \leq r < 1}{\text{g.l.b.}}\ B(\alpha,r) = B(\alpha,0) = 1$. In case $\alpha = \frac{1}{2}$, it is obvious.

Hence (4.37) holds. This completes the proof of the theorem. The result is sharp as it can be seen from the functions

$$F(z) = \frac{z}{(1-z)^{2(1-\beta)}}$$

and

$$g(z) = \begin{cases} \frac{1}{2\alpha-1}(1+z)^{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ \log(1+z), & \alpha = \frac{1}{2}. \end{cases}$$

In this theorem if we put $\alpha = \beta = 0$ and $c = 0$ we get the theorem 1 of Calys [11] as a corollary.

Theorem 4.2.4. If $f(z)$ is regular in D and satisfy (2.33) where $F \in S^*(\beta)$ and $g(z)/z \in P(\alpha)$ then $f(z)$ is univalent and starlike of order β in $|z| < r_0$ where r_0 is the smallest positive root of the equation

$$(4.38) \quad (c+2)(1-\beta)-2\{(c+2)(1-\alpha\beta)+(1-\beta)(2-\beta)\}\, r - 2\,\{c(3-4\alpha-\beta+\alpha\beta) + (3+2\beta-8\alpha+6\alpha\beta-\beta^2-2\alpha\beta^2)\}r^2+2\{(c+2\beta)(2\alpha-\alpha\beta-1) - (2\alpha-1) \times (1-\beta)(2-\beta)\}\, r^3 - (2\alpha-1)(1-\beta)(c+2\beta)\, r^4 = 0$$

The result is sharp.

Proof : Since $\frac{g(z)}{z} \in P(\alpha)$, therefore from a result of Libera [29] we have

$$(4.39) \qquad \mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \leq \frac{1-2(2\alpha-1)r+(2\alpha-1)r^2}{(1-r)(1-(2\alpha-1)r)}.$$

Proceeding on the same lines as in theorem 4.2.2 we get

(4.40) $$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\} > 0$$

if

(4.41) $$1 - \frac{1-2(2\alpha-1)r+(2\alpha-1)r^2}{(1-r)(1-(2\alpha-1)r)} + \frac{(1-\beta)\left\{1 - \frac{2(2-\beta)}{c+2}r - \frac{c+2\beta}{c+2}r^2\right\}}{(1+r)\left\{1 + \frac{c+2\beta}{c+2}r\right\}} > 0$$

and

(4.42) $$P(r(\beta)) = 1 - \frac{1 - 2(2\alpha-1)r(\beta) + (2\alpha-1)\, r^2(\beta)}{(1 - r(\beta))(1-(2\alpha-1)r(\beta))} < 0$$

From (4.40) and (4.41) we get (4.38). So proof will be complete if we prove (4.42). Now from (4.42) we have

$$P(r(\beta)) = - \frac{2(1-\alpha)\, r(\beta)}{(1-r(\beta))\,\{1-(2\alpha-1)r(\beta)\}}$$

$$< 0.$$

The result is sharp as can be seen from the following functions

$$F(z) = \frac{z}{(1-z)^{2(1-\beta)}}$$

and

$$g(z) = \frac{z\,\{1 + (2\alpha-1)z\}}{1+z)}$$

In this theorem if we take $c = \alpha = \beta = 0$ we get the result (theorem 3) of Calys [11].

Theorem 4.2.5 : Let $F \in \Gamma(m,M)$ and $f(z)$ be defined by (2.40) and $r(a,b)$ be the unique positive root of the equation

$$(4.43) \quad (a+d)+2\{d(a+b)-(d-b)\}r+\{2(b^2-d^2)-(a+d)-d(ad+b^2)\}r^2$$

$$-2d\{(a+b)+b(d-b)\}r^3-d(ad+b^2)r^4=0$$

and $d \leq 0$ then $f(z)$ is meromorphic starlike of order β for $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(4.44) \qquad (1-\beta)+\{(a+b+2d)-(b+d)\beta\}r+(ab+bd+d^2-bd\beta)r^2=0$$

if $0 < r_0 \leq r(a,b)$, and that of the equation

$$(4.45) \quad (E-1+bd)-(1+bd)x+\sqrt{(1+d)\{(1+d)+(1-d)x\}\{(1-2a+b^2)+(1-b^2)x\}}$$

if $r(a,b) \leq r_0$ where

$$x=\frac{1+r^2}{1-r^2}, \quad E=(a-b)-(d-b)\beta \quad \text{and } d=\frac{a+b+bc}{c}$$

The result is sharp.

Proof : From (2.40) we have

$$(4.46) \qquad c\,\frac{f(z)}{F(z)}=(c+1)+\frac{zF'(z)}{F(z)}.$$

Since $F \in \Gamma(m,M)$ there exists a regular function $w(z)$ with $w(0)=0$, $|w(z)| < 1$ and

$$(4.47) \qquad \frac{zF'(z)}{F(z)} = -\frac{1 + aw(z)}{1 - bw(z)}$$

From (4.46) and (4.47) we have

$$(4.48) \qquad \frac{f(z)}{F(z)} = \frac{1 - \frac{a+b+bc}{c}\, w(z)}{1 - bw(z)}$$

$$= \frac{1 - dw(z)}{1 - bw(z)} \, .$$

Differentiating (4.48) logarithmically with respect to z and using (4.47), we get,

$$\frac{zf'(z)}{f(z)} = -\frac{1+aw(z)}{1-bw(z)} - \frac{dz\,w'(z)}{1-dw(z)} + \frac{bz\,w'(z)}{1-bw(z)}$$

or

$$(4.49) \quad \frac{zf'(z)}{f(z)} + \beta = \beta - \frac{1+aw(z)}{1-bw(z)} - \frac{a+b}{c} \cdot \frac{w(z)}{(1-dw(z))(1-bw(z))}$$

$$- \frac{a+b}{c} \cdot \frac{zw'(z)-w(z)}{(1-dw(z))(1-bw(z))} \, .$$

Using the well known inequality $|zw'(z)-w(z)| \leq \dfrac{r^2-|w(z)|^2}{1-r^2}$ we get

$$(4.50) \quad -\mathrm{Re}\,\left\{\frac{zf'(z)}{f(z)} + \beta\right\} \geq -\beta + \mathrm{Re}\,\left\{\frac{1+aw(z)}{1-bw(z)}\right\} +$$

$$+ \frac{a+b}{c} \cdot \mathrm{Re}\, \frac{w(z)}{(1-dw(z))(1-bw(z))} - \frac{a+b}{c} \cdot \frac{(r^2-|w(z)|^2)}{(1-r^2)|1-dw(z)||1-bw(z)|}$$

Let us take

$$(4.51) \qquad p(z) = \frac{1-dw(z)}{1-bw(z)} \, .$$

From (4.50) and (4.51), we get,

$$(4.52) \quad -\mathrm{Re}\,\{\frac{zf'(z)}{f(z)} + \beta\} \geq \frac{c}{a+b}\,[E - a\,\mathrm{Re}\,\{p(z)\} + d\,\mathrm{Re}\,\{\frac{1}{p(z)}\} - $$

$$- \frac{r^2|d-bp(z)|^2 - |1-p(z)|^2}{(1-r^2)\,|p(z)|}].$$

From (4.51) it is easy to see that

$$(4.53) \qquad |p(z) - A| \leq B$$

where

$$(4.54) \qquad A = \frac{1-bd\ r^2}{1-b^2r^2}$$

and

$$(4.55) \qquad B = \frac{(d-b\ r}{1-b^2r^2}\ .$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$ and use (4.53), (4.54) and (4.55) we can rewrite (4.52) as

$$(4.56) \quad -\mathrm{Re}\,\{\frac{zf'(z)}{f(z} + \beta\} \geq \frac{c}{a+b}\,[E - a(A+u) + \frac{d(A+u)}{R^2}$$

$$- \frac{B^2-u^2-v^2}{R}\,(\frac{1-b^2r^2}{1-r^2})]$$

$$\equiv \frac{c}{a+b}\,P(u,v).$$

Differentiating P(u,v) partially with respect to v we have

$$\frac{\partial P(u,v)}{\partial v} = \frac{v}{R}\left[-\frac{2d(A+u)}{R^3} + \left\{2 + \frac{B^2-u^2-v^2}{R^2}\right\}\left(\frac{1-b^2r^2}{1-r^2}\right)\right].$$

If we take $d \leq 0$ then $\frac{\partial P(u,v)}{\partial v} > 0$ if $v > 0$ and $\frac{\partial P(u,v)}{\partial v} < 0$ if $v < 0$

hence

$$\min_v P(u,v) = P(u,0)$$

$$= E - aR + \frac{d}{R} - \frac{B^2 - (A-R)^2}{R} \cdot \left(\frac{1-b^2r^2}{1-r^2}\right)$$

$$\equiv P(R)$$

where $R = A + u$. Now

$$P'(R) = -a - \frac{d}{R^2} - \left(\frac{A^2-B^2}{R^2} - 1\right)\left(\frac{1-b^2r^2}{1-r^2}\right)$$

and

$$P''(R) = \frac{2}{R^3}\left[d + (A^2-B^2)\left(\frac{1-b^2r^2}{1-r^2}\right)\right]$$

$$= \frac{2(1+d)(1-dr^2)}{R^3(1-r^2)}$$

$$\geq 0.$$

Therefore $P'(R)$ is an increasing function of R and $P(R_o) = 0$ where

(4.57) $$R_o = \left[\frac{(1+d)(1-dr^2)}{(1-a) + (a-b^2)r^2}\right]^{1/2}.$$

Since $P'(A-B) \leq 0$ and $P'(R)$ is an increasing function of R so $A-B \leq R_o$.

So

$$(4.58) \quad \min_R P(R) = \begin{cases} P(A+B) & \text{if } A+B \leq R_o \\ P(R_o) & \text{if } A+B \geq R_o. \end{cases}$$

$$= \begin{cases} E - a\left(\frac{1+dr}{1+br}\right) + d\left(\frac{1+br}{1+dr}\right) & \text{if } A+B \leq R_o \\ E - \frac{2\ 1-bdr^2)}{(1-r^2)} + \frac{2}{R_o} \cdot \frac{(1+d)(1-dr^2)}{1-r^2} & \text{if } A+B \geq R_o \end{cases}$$

$$= \begin{cases} \frac{(d-b)[(1-\beta+\{a+b+2d-(b+d)\beta\}r+(ab+bd+d^2-bd\beta)r^2]}{(1+br)\,[redacted]} & \text{if } A+B \leq R_o \\ (E-1+bd)-(1+bd)x+\sqrt{(1+d)\{(1+d)+(1-d)x\}\{(1-2a+b^2)+(1-b^2)x\}} & \text{if } A+B \geq R_o. \end{cases}$$

where $x = \frac{1+r^2}{1-r^2}$.

Let us take

$$(4.59) \quad Q(r) = (A+B)^2 - R_o^2$$

$$= \left(\frac{1+dr}{1+br}\right)^2 - \left[\frac{(1+d)\ (1-dr^2)}{(1-a)+(a-b^2)r^2}\right]$$

$$= \frac{-[(a+d)+2\{d(a+b)-(d-b)\}r+\{2(b^2-d^2)-(a+d)+d(ad+b^2)\}r^2 - 2d\{(a+b)+b(d-b)\}r^3 - d(ad+b^2\ \ r^4]}{(1+br)^2\{(1-a)+(a-b^2)r^2\}}$$

$$Q'(r) = \frac{2(1+dr)(d-b)}{(1+br)^3} + \frac{2(1+d)(a+b)\ \{(1-a)+c(1-b)\}r}{\{(1-a)+(a-b^2)r^2\}^2}$$

$$\geq 0.$$

$$Q(0) = -\frac{a+d}{1-a} < 0$$

$$Q(1) = \frac{2(1+d)(d-b)}{(1+b)^2(1-b)} > 0.$$

Since $Q(r)$ is an increasing function of r and $Q(0) < 0$ and $Q(1) > 0$ so there is only one root of the equation $Q(r) = 0$. Let this root be $r(a,b)$. So if $r \leq r(a,b)$, $(A+B) \leq R_0$ and if $r > r(a,b)$, $A+B > R_0$.

Therefore result follows from (4.58) and (4.59). Equality in (4.44) is attained for the function

$$F(z) = \frac{(1+bz)^{\frac{a+b}{b}}}{z}$$

and that in (4.45) for the function

$$F(z) = \frac{[(1-bz)^{1+k}(1+bz)^{1-k}]^{\frac{a+b}{2b}}}{z}$$

where k is determined from

$$\frac{1-k(a+b)z + ab\, z^2}{1-b^2z^2} = \left[\frac{(1+d)(1-dr^2)}{(1-a)+(a-b^2)r^2}\right]^{1/2}$$

CHAPTER - 5

A GENERALIZATION OF FUNCTION WITH BOUNDED BOUNDARY ROTATION

5.1 Work of this chapter is motivated by investigations carried out by authors like E.J. Moulis [38] , R.J. Leach [27] and others. The class V_k of functions of bounded boundary rotation was introduced by K. Lowner [33] and most of the nice results in this direction are due to V. Paatero [44,45] , O. Lehto [28] , W.E. Kirwan [25] , M.S. Robertson [56] , D.A. Brannan [9] , B. Pinchuk [49] , J.W. Noonan [41,42] , D.A. Brannan, J.G. Clunie and W.E. Kirwan [10], and many other mathematicians. In the present chapter our aim is to generalize the classes investigated by Moulis [38] . However the study of these generalized classes is quite difficult. In fact none of the classical methods yield powerful results. This has been seen even in the less generalized classes due to Leach [27] . We study the following class :

<u>Definition</u> : $V_\alpha(k,p)$ denotes the class of functions satisfying the conditions :

(i) $f(z)$ is analytic in D.

(ii) $f(0) = 0$, $f'(0) = 1$ and $f(ze^{2\pi i/p})=e^{2\pi i/p} f(z)$, p is a positive int

(iii) $f'(z) \neq 0$ in D.

and

(iv)

$$(5.1) \qquad \int_0^{2\pi} \left|\mathrm{Re}\left\{e^{i\alpha}\left[1+\frac{zf''(z)}{f'(z)}\right]\right\}\right| d\theta \leq k\pi\cos\alpha$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $k \geq 2$ and $|\alpha| < \pi/2$.

Functions in the class $V_\alpha(k,p)$ with $\alpha \neq 0$, do not necessarily have bounded boundary rotation still they have many properties similar to those of the functions with bounded boundary rotation. Functions in $V_\alpha(2,p)$ satisfy

$$(5.2) \qquad \int_0^{2\pi} \left|\operatorname{Re} \{e^{i\alpha} [1 + \frac{zf''(z)}{f'(z)}]\}\right| d\theta = 2\pi \cos \alpha$$

and an argument based on the continuity of the integrand in (5.2) shows that we must have

$$\operatorname{Re} \{e^{i\alpha} [1 + \frac{zf''(z)}{f'(z)}]\} = \operatorname{Re} \{e^{i\alpha} [\frac{z(zF'(z))'}{zF'(z)}]\} > 0, \; z \in D$$

which means $zf'(z)$ is α-spirallike in D.

5.2 In this section we state some results as lemmas which we shall use in proving our results in following sections.

Lemma 5.2.1. Let $s(z) \in S^*$. Then for $|z| \leq r$, $|\arg \{\frac{s(z)}{z}\}| \leq 2 \operatorname{arc} \sin r$ and this result is sharp.

Proof of this result can be found in [18] .

Lemma 5.2.2 : Let $S(z)$ be p-fold symmetric starlike function. Then for $|z| \leq r$,

$$(5.3) \qquad |\arg \{\frac{S(z)}{z}\}| \leq \frac{2 \operatorname{arc} \sin r^p}{p}$$

and this result is sharp.

Proof : Let us define $s(z)$ such that $S(z) = [s(z^p)]^{1/p}$. On logarithmic differentiation with respect to z this gives

$$\frac{zS'(z)}{S(z)} = z^p \frac{s'(z^p)}{s(z^p)} .$$

Since $S(z)$ is p-fold symmetric starlike function, $s(z) \in S^*(0)$. Now by lemma 5.2.1 we get

$$\left|\arg \left\{\frac{S(re^{i\theta})}{re^{i\theta}}\right\}\right| = \left|\arg \left\{\frac{s(r^p e^{pi\theta})}{r^p e^{pi\theta}}\right\}^{1/p}\right|$$

$$\leq \frac{2 \arcsin r^p}{p} .$$

It can be easily seen that this result is sharp because equality holds for the function

$$S(z) = \frac{z}{(1-z^p)^2} .$$

Lemma 5.2.3 : Let $q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be regular in D. Then

$$(5.4) \qquad e^{i\alpha} q(z) = \frac{\cos \alpha}{2\pi} \int_0^{2\pi} \frac{1 + ze^{i\phi}}{1 - ze^{i\phi}} \, d\psi(\theta) + i \sin \alpha$$

where $\psi(\phi)$ is a function of bounded variation on $[0,2\pi]$ satisfying

$$\int_0^{2\pi} d\psi(\phi) = 2\pi$$

and

$$\int_0^{2\pi} |d\psi(\phi)| = \lim_{r \to 1} \int_0^{2\pi} \frac{\operatorname{Re} \{e^{i\alpha} q(re^{i\theta})\}}{\cos \alpha} \, d\theta.$$

__Proof__ : Let us define $s(z)$ such that $S(z) = [s(z^p)]^{1/p}$. On logarithmic differentiation with respect to z this gives

$$\frac{zS'(z)}{S(z)} = z^p \frac{s'(z^p)}{s(z^p)} .$$

Since $S(z)$ is p-fold symmetric starlike function, $s(z) \in S^*(0)$. Now by lemma 5.2.1 we get

$$\left|\arg \left\{\frac{S(re^{i\theta})}{re^{i\theta}}\right\}\right| = \left|\arg \left\{\frac{s(r^p e^{pi\theta})}{r^p e^{pi\theta}}\right\}^{1/p}\right|$$

$$\leq \frac{2 \text{ arc sin } r^p}{p} .$$

It can be easily seen that this result is sharp because equality holds for the function

$$S(z) = \frac{z}{(1-z^p)^2} .$$

__Lemma 5.2.3__ : __Let__ $q(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ __be regular in__ D. __Then__

$$(5.4) \qquad e^{i\alpha} q(z) = \frac{\cos \alpha}{2\pi} \int_0^{2\pi} \frac{1 + ze^{i\phi}}{1 - ze^{i\phi}} d\psi(\theta) + i \sin \alpha$$

__where__ $\psi(\phi)$ __is a function of bounded variation on__ $[0,2\pi]$ __satisfying__

$$\int_0^{2\pi} d\psi(\phi) = 2\pi$$

__and__

$$\int_0^{2\pi} |d\psi(\phi)| = \lim_{r \to 1} \int_0^{2\pi} \frac{\text{Re } \{e^{i\alpha} q(re^{i\theta})\}}{\cos \alpha} d\theta.$$

Proof of this lemma is due to Moulis [38,page 17] .

<u>Lemma 5.2.4</u> : <u>Let</u> $g(z) = \sum_{n=0}^{\infty} b_n z^n \in V_k$ <u>and suppose that</u>

$$f'(z) = \exp \{-\frac{1}{\pi} \int_0^{2\pi} \log (1-ze^{-i\phi}) \, d\mu \, \phi\}$$

<u>then for</u> $n \geq 2$

$$(5.5) \qquad b_n = \frac{1}{n(n-1)} \sum_{j=1}^{n-1} j \, b_j \frac{1}{\pi} \int_0^{2\pi} e^{-(n-j)i\phi} \, d\mu(\phi) .$$

Proof of this result is due to Lehto [28] .

5.3 In this section we prove some representation theorems for the class $V_\alpha(k,p)$.

<u>Theorem 5.3.1.</u> $f \in V_\alpha(k,p)$ <u>if and only if there exists a function</u> $g \in V_\alpha(k,1)$ <u>such that</u>

$$(5.6) \qquad f'(z) = [g'(z^p)]^{1/p} .$$

<u>Proof</u> : Let $g(z) \in V_\alpha(k,1)$. Since $g'(z) \neq 0$ in D, $f(z) = \int_0^z [g'(t^p)]^{1/p}$ is single valued and analytic in D. Since $[f'(z)]^p = g'(z^p)$ we have

$$1 + \frac{z^p \, g''(z^p)}{g'(z^p)} = 1 + \frac{zf''(z)}{f'(z)} .$$

Therefore,

Proof of this lemma is due to Moulis [38, page 17] .

Lemma 5.2.4 : Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in V_k$ and suppose that

$$f'(z) = \exp \{-\frac{1}{\pi} \int_0^{2\pi} \log (1-ze^{-i\phi}) \, d\mu \, \phi\}$$

then for $n \geq 2$

$$(5.5) \qquad b_n = \frac{1}{n(n-1)} \sum_{j=1}^{n-1} j \, b_j \frac{1}{\pi} \int_0^{2\pi} e^{-(n-j)i\phi} \, d\mu(\phi) \, .$$

Proof of this result is due to Lehto [28] .

5.3 In this section we prove some representation theorems for the class $V_\alpha(k,p)$.

Theorem 5.3.1. $f \in V_\alpha(k,p)$ if and only if there exists a function $g \in V_\alpha(k,1)$ such that

$$(5.6) \qquad f'(z) = [g'(z^p)]^{1/p} \, .$$

Proof : Let $g(z) \in V_\alpha(k,1)$. Since $g'(z) \neq 0$ in D, $f(z) = \int_0^z [g'(t^p)]^{1/p}$ is single valued and analytic in D. Since $[f'(z)]^p = g'(z^p)$ we have

$$1 + \frac{z^p \, g''(z^p)}{g'(z^p)} = 1 + \frac{zf''(z)}{f'(z)} \, .$$

Therefore,

$$\int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \frac{zf''(z)}{f'(z)}]\}|\ d\theta = \int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \frac{z^p g''(z^p)}{g'(z^p)}]\}|\ d\theta, \quad z=re^{i\theta}$$

$$= \int_0^{2\pi} |\mathrm{Re}\ \ e^{i\alpha}\ [1 + \frac{tg''(t)}{g'(t)}]\}|\ d\phi\ , \qquad t=r^p e^{i\phi}$$

Thus

$$(5.7)\quad \lim_{|z| \to 1} \int_0^{2\pi} |\mathrm{Re}\ \ e^{i\alpha}\ [1 + \frac{zf''(z)}{f'(z)}]\}|\ d\theta$$

$$= \lim_{|t| \to 1} \int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \frac{tg''(t)}{g'(t}]\}|\ d\phi$$

Since $f(z)$ is p-fold symmetric whenever $g \in V_\alpha(k,1)$ therefore from (5.7) we have $f \in V_\alpha(k,p)$ if and only if $g \in V_\alpha(k,1)$. This completes the proof of the theorem.

<u>Theorem 5.3.2.</u> <u>Let</u> $f \in V_o(k,p)$. <u>Then</u> $F(z) = \int_0^z \frac{f(t)}{t}\,dt \in V_o(k,p)$.

<u>Proof</u> : It is obvious that F is p-fold symmetric. From the given relation we get

$$1 + \frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)}\ .$$

Therefore

$$(5.8)\qquad \int_0^{2\pi} |\mathrm{Re}\ \{1 + \frac{zF''(z)}{F'(z)}\}|\ d\theta = \int_0^{2\pi} |\mathrm{Re}\ \{\frac{zf'(z)}{f(z)}\}|\ d\theta\ ,\ z = re^{i\theta},$$

Biernacki [8] has shown that if $f(z)$ is analytic in D, then

$$(5.9)\qquad \int_0^{2\pi} |\mathrm{Re}\ \{\frac{zf'(z)}{f(z)}\}|\ d\theta \leq \int_0^{2\pi} |\mathrm{Re}\ \{1 + \frac{zf''(z)}{f'(z)}\}|\ d\theta.$$

From (5.8) and (5.9), we get,

$$\int_0^{2\pi} \left|\text{Re}\ \left\{1 + \frac{zF''(z)}{F'(z)}\right\}\right| d\theta \le \int_0^{2\pi} \left|\text{Re}\ \left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta$$

$$\le k\pi$$

and hence $F \in V_0(k,p)$.

Corollary 5.3.1. : Let $f \in V_0(k,1)$. Then $F(z) = \int_0^z \frac{[f(t^p)]^{1/p}}{t} dt \in V_0(k,p)$

Proof : Since $f \in V_0(k,p)$ therefore from theorem 5.3.1 we get $[f(z^p)]^{1/p} \in V_0(k,p)$ and hence from theorem 5.3.2 the result follows.

The results in next three theorems can be obtained by using theorem 5.3.1 and the corresponding results due to Moulis [38]. But for the completeness we give here their proofs.

Theorem 5.3.3. If $f \in V_\alpha(k,1)$ and $F(z)$ is defined by

$$(5.10) \qquad F'(z) = \left[\frac{f'\left(\frac{z^p+a}{1+\bar{a}z^p}\right)}{f'(a)(1+\bar{a}z^p)^{(1-e^{-2i\alpha})}}\right]^{1/p}$$

where $z \in D$, $|a| < 1$ and $F(0) = 0$ then $F(z) \in V_\alpha(k,p)$.

Proof : Let ρ be a real number in $(0,1)$ and a be any complex number with $|a| < 1$ for $f(z)$ in $V_\alpha(k,1)$ define $F_\rho(z)$ by the equation

$$(5.11) \qquad F_\rho'(z) = \left[\frac{f'(\rho t)}{f'(\rho a)(1+\bar{a}z^p)^{1+e^{-2i\alpha}}}\right]^{1/p}$$

where

$$t = \frac{a + z^p}{a + \bar{a}z^p}, \quad F_\rho(0) = 0.$$

Differentiating logarithmically the function $F'_\rho(z)$ with respect to z, w get,

$$\frac{F''_\rho(z)}{F'_\rho(z)} = \frac{1}{p}\left[\frac{\rho f''(\rho t)}{f'(\rho t)} \cdot \frac{p(1-|a|^2)z^{p-1}}{(1+\bar{a}\, z^p)^2} - \frac{(1+e^{-2i\alpha})\bar{a}pz^{p-1}}{1+\bar{a}\, z^p}\right]$$

or

$$1 + \frac{zF''_\rho(z)}{F'_\rho(z)} = \{1 + \frac{\rho t\, f''(\rho t)}{f'(\rho t)}\} \cdot \frac{(1-|a|^2)\, z^p}{(1+\bar{a}z^p)(a+z^p)}$$

$$+ \frac{|a\,(1-e^{-2i\alpha})\, z^p + a - \bar{a}\, e^{-2i\alpha}\, z^{2p}}{(1+\bar{a}\, z^p)(a+z^p)}$$

or

$$(5.12) \qquad \mathrm{Re}\,\{e^{i\alpha}\,[1 + \frac{z\, F''_\rho(z)}{F'_\rho(z)}]\} = \mathrm{Re}\,\{e^{i\alpha}\,[1 + \frac{\rho t\, f''(\rho t)}{f'(\rho t)}] \cdot \frac{(1-|a|^2)z^p}{(1+\bar{a}z^p)(a+z^p)}\}$$

$$+ \mathrm{Re}\{\frac{2|a|^2\ i\sin\alpha\, z^p + ae^{i\alpha} - \bar{a}e^{-i\alpha}\ z^{2p}}{(1+\bar{a}\, z^p)(a+z^p)}\}$$

It we set $z = e^{i\theta}$

$$(5.13) \qquad \mathrm{Re}\,\{\frac{2|a|^2\ i\sin\alpha.\ z^p + ae^{i\alpha} - \bar{a}\, e^{-i\alpha}\ z^{2p}}{(1+\bar{a}\, z^p)(a+z^p)}\}$$

$$= \mathrm{Re}\,\{\frac{2|a|^2\ i\sin\alpha + \frac{ae^{i\alpha}}{z^p} - \bar{a}\, e^{-i\alpha}z^p}{\frac{a}{z^p} + |a|^2 + \bar{a}\, z^p}\}$$

$$= \mathrm{Re}\,\{\frac{2\,|a|^2\ i\sin\alpha + ae^{i\alpha}\, e^{-pi\theta} - \bar{a}\, e^{-i\alpha}e^{pi\theta}}{(ae^{-pi\theta} + \bar{a}\, e^{pi\theta}) + |a|^2}\}$$

Thus if we have $z = re^{i\theta}$ and

$$(5.14) \qquad \frac{a + e^{pi\theta}}{1+\bar{a}\ e^{pi\theta}} = e^{pi\phi}$$

then from (5.12) and (5.13) we get

$$(5.15) \quad \mathrm{Re}\ \{e^{i\alpha}\ [1 + e^{i\theta}\ \frac{F_\rho''(e^{i\theta})}{F_\rho'(e^{i\theta})}]\} = \mathrm{Re}\ \{e^{i\alpha}\ [1 + \rho e^{pi\phi}\ \frac{f''(\rho e^{pi\phi})}{f'(\rho e^{pi\phi})}]\ \cdot$$

$$\frac{(1-|a|^2)\ e^{pi\theta}}{(1 + \bar{a}\ e^{pi\theta})(a + e^{pi\theta})}\}$$

Differentiating (5.14) we have

$$(5.16) \qquad d\theta = \frac{|a + e^{pi\theta}|^2}{1 - |a|^2} \cdot d\phi$$

From (5.15) and (5.16) we have

$$\mathrm{Re}\ \{e^{i\alpha}\ [1 + e^{i\theta}\ \frac{F_\rho''(e^{i\theta})}{F_\rho'(e^{i\theta})}]\}\ d\theta = \mathrm{Re}\ \{e^{i\alpha}\ [1 + \rho e^{pi\phi}\ \frac{f''(\rho e^{pi\phi})}{f'(\rho e^{pi\phi})}\}]d\phi$$

or

$$(5.17) \quad \int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + e^{i\theta}\ \frac{F_\rho''(e^{i\theta})}{F_\rho'(e^{i\theta})}]\}|\ d\theta$$

$$= \int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \rho e^{pi\phi}\ \frac{f''(\rho e^{pi\phi})}{f'(\rho e^{pi\phi})}]\}|\ d\phi$$

$$= \int_0^{2p\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \rho e^{i\psi}\ \frac{f''(\rho e^{i\psi})}{f'(\rho e^{i\psi})}]\}|\ \frac{d\psi}{p}$$

$$= \int_0^{2\pi} |\mathrm{Re}\ \{e^{i\alpha}\ [1 + \rho e^{i\psi}\ \frac{f''(\rho e^{i\psi})}{f'(\rho e^{i\psi})}]\}|\ d\psi$$

$$\leq k\pi \quad \cos\alpha\ .$$

The integral

$$(5.18) \qquad I_F(r) = \int_0^{2\pi} |\mathrm{Re}\,\{e^{i\alpha}\,[1 + re^{i\theta}\,\frac{F_\rho''(re^{i\theta})}{F_\rho'\,(re^{i\theta})}]\}|\,d\theta$$

is an increasing function of r, $0 \leq r < 1$ because its integrand is the absolute value of a harmonic function therefore

$$(5.19) \qquad I_F(r) \leq \lim_{r \to 1} I_F(r)$$

From (5.17), (5.18) and (5.19) we get

$$(5.20) \qquad I_F(r) \leq k\pi \cos\alpha .$$

From (5.20) and the fact that F(z) is p fold symmetric we get the result.

Theorem 5.3.4 : $f \in V_\alpha(k,p)$ if and only if there exists a function $\psi(\theta)$ of bounded variation on $[0,2\pi]$ such that

$$(5.21) \qquad f(z) = \int_0^z \exp\{-\frac{e^{-i\alpha}\cos\alpha}{p\pi}\int_0^{2\pi} \log\,(1-t^p e^{i\theta})\,d\psi(\theta)\}\,dt$$

where $\psi(\theta)$ is normalized by the condition

$$(5.22) \qquad \int_0^{2\pi} d\psi(\theta) = 2\pi$$

and satisfies the condition

$$(5.23) \qquad \int_0^{2\pi} d\psi(\theta)| \leq k\pi$$

<u>Proof</u> : If $g(z) \in V_o(k,1)$ then using a formula due to Paatero [44] we get

$$(5.24) \qquad g'(z) = \exp\{-\frac{1}{\pi}\int_0^{2\pi} \log(1-ze^{i\theta})\, d\psi(\theta)\}$$

where $\psi(\theta)$ satisfies the conditions (5.22) and (5.23). Differentiating $g'(z)$ logarithmically with respect to z we have

$$(5.25) \qquad 1 + \frac{zg''(z)}{g'(z)} = 1 + \frac{1}{\pi}\int_0^{2\pi} \frac{e^{i\theta}}{1-ze^{i\theta}}\, d\psi(\theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\psi(\theta) + \frac{1}{2\pi}\int_0^{2\pi} \frac{2e^{i\theta}}{1-ze^{i\theta}}\, d\psi(\theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\, d\psi(\theta).$$

If we take $q(z^p) = 1 + \frac{zf''(z)}{f'(z)}$ in lemma 5.2.3 we get

$$(5.26) \qquad e^{i\alpha}\{1 + \frac{zf''(z)}{f'(z)}\} = \frac{\cos\alpha}{\pi}\int_0^{2\pi} \frac{1+z^p e^{i\theta}}{1-z^p e^{i\theta}}\, d\psi(\theta) + i\sin\alpha.$$

From (5.25) and (5.26) we get

$$e^{i\alpha}\{1 + \frac{zf''(z)}{f'(z)}\} = \cos\alpha\,\{1 + \frac{z^p g''(z^p)}{g'(z^p)}\} + i\sin\alpha$$

or

$$\frac{f''(z)}{f'(z)} = e^{-i\alpha}\cos\alpha\,\{\frac{1}{z} + \frac{z^{p-1}\, g''(z^p)}{g'(z^p)}\} + \frac{ie^{-i\alpha}\sin\alpha - 1}{z}$$

$$= e^{-i\alpha}\cos\alpha \cdot \frac{z^{p-1}\, g''(z^p)}{g'(z^p)}$$

or

$$\log f'(z) = \frac{e^{-i\alpha} \cos\alpha}{p} \log g'(z^p)$$

or

$$(5.27) \qquad f'(z) = [g'(z^p)]^{\frac{e^{-i\alpha}\cos\alpha}{p}} .$$

From (5.24) and (5.27) the result follows.

<u>Theorem 5.3.5</u> : $f(z) \in V_\alpha(k,p)$ <u>if and only if there are two p-fold symmetric starlike functions</u> $S_1(z)$ <u>and</u> $S_2(z)$ <u>such that</u>

$$(5.28) \qquad f'(z) = \left\{\left[\frac{S_1(z)}{z}\right]^{\frac{k+2}{4}} \Big/ \left[\frac{S_2(z)}{z}\right]^{\frac{k-2}{4}}\right\}^{\frac{e^{-i\alpha}\cos\alpha}{p}}$$

<u>Proof</u> : Brannan [9] has shown that $g(z) \in V_o(K,1)$ if and only if there are two starlike functions $s_1(z)$ and $s_2(z)$ such that

$$(5.29) \qquad g'(z) = \left\{\left[\frac{s_1(z)}{z}\right]^{\frac{k+2}{4}} \Big/ \left[\frac{s_2(z)}{z}\right]^{\frac{k-2}{4}}\right\} .$$

Using the representation in (5.27) and the relation (5.29), we get,

$$f'(z) = \left\{\left[\frac{s_1(z^p)}{z^p}\right]^{\frac{k+2}{4}} \Big/ \left[\frac{s_2(z^p)}{z^p}\right]^{\frac{k-2}{4}}\right\}^{\frac{e^{-i\alpha}\cos\alpha}{p}}$$

$$= \left\{\left[\frac{s_1(z^p)^{1/p}}{z}\right]^{\frac{k+2}{4}} \Big/ \left[\frac{s_2(z^p)^{1/p}}{z}\right]^{\frac{k-2}{4}}\right\}^{e^{-i\alpha}\cos\alpha} .$$

If we take $S_1(z) = [s_1(z^p)]^{1/p}$ and $S_2(z) = [s_2(z^p)]^{1/p}$ then $S_1(z)$ and $S_2(z)$ are obviously p-fold symmetric and starlike in D. This completes the proof of the theorem.

Corollary 5.3.2 : $f \in V_\alpha(k,p)$ if and only if

(5.30) $$f'(z) = \{[\frac{T_1(z)}{z}]^{\frac{k+2}{4}} / [\frac{T_2(z)}{z}]^{\frac{k-2}{4}}\}$$

where $T_1(z)$ and $T_2(z)$ are normalized p-fold symmetric α-spiral functions.

Proof : If $S(z)$ is starlike then

(5.31) $$T(z) = z\,[\frac{S(z)}{z}]^{e^{-i\alpha}\cos\alpha}$$

is α-spiral, since

$$\frac{zT'(z)}{T(z)} = 1 + e^{-i\alpha}\cos\alpha\,[\frac{zS'(z)}{S(z)} - 1]$$

$$= e^{-i\alpha}\,[\cos\alpha \cdot \frac{zS'(z)}{S(z)} + e^{i\alpha} - \cos\alpha]$$

or

(5.32) $$\mathrm{Re}\,\{e^{i\alpha}\,\frac{zT'(z)}{T(z)}\} = \cos\alpha\,\mathrm{Re}\,\{\frac{zS'(z)}{S(z)}\}$$

$$> 0.$$

Again since

$$T(ze^{2\pi i/p}) = ze^{2\pi i/p}\,[\frac{S(ze^{2\pi i/p})}{ze^{2\pi i/p}}]^{e^{-i\alpha}\cos\alpha}$$

$$= e^{2\pi i/p}\,T(z)$$

the result follows from (5.32).

5.4 In this section we have obtained restriction on α such that functions of class $V_\alpha(k,p)$ become univalent and also obtained the discs

in which $f(z)$ is univalent, convex and $zf'(z)$ is α-spiral function. We have also obtained some distortion theorems.

Theorem 5.4.1 : If $g(z)$ is in $V_\alpha(k,p)$ then $g(z)$ is univalent in the disc $|z| < r_o$ where r_o is the smallest positive root of the equation

$$(5.33) \qquad 1 - k \cos\alpha \, . \, r^{p-1} - 2\cos\alpha \, . \, r^{2p-1} = 0 \, .$$

Proof : Differentiating logarithmically the function $F(z)$ in (5.10) with respect to z, we have,

$$\frac{F''(z)}{F'(z)} = \frac{f''(t)}{f'(t)} \, . \, \frac{(1-|a|^2 \, z^{p-1}}{(1+\bar{a}z^p)^2} - \frac{(e^{-2i\alpha}+1 \;\; \bar{a} \; z^{p-1}}{(1+\bar{a} \; z^p)} \; , \; t = \frac{a+z^p}{1+\bar{a}z^p}$$

or

$$\frac{F''(z)}{z^{p-1}} = [\frac{f''(t)}{f'(t)} \, . \, \frac{(1-|a|^2)}{(1+\bar{a}z^p)^2} - \frac{(e^{-2i\alpha}+1)\bar{a}}{1+\bar{a}z^p}] \; F'(z).$$

Taking limit as $z \to 0$ we have

$$(5.34) \qquad p(p+1) \, a_{p+1} = \frac{f''(a)}{f'(a)} \, (1-|a|^2) - (e^{-2i\alpha}+1) \, \bar{a}$$

where a_{p+1} is the coefficient of z^{p+1} in the expansion of $F(z)$. From this relation we get

$$(5.35) \qquad \left|\frac{f''(a)}{f'(a)} - \frac{(e^{-2i\alpha}+1)\bar{a}}{1-|a|^2}\right| = \frac{p(p+1) \, |a_{p+1}|}{1-|a|^2}$$

$$\leq \frac{k \cos\alpha}{1-|a|^2} \, .$$

Here we have used the relation $|a_{p+1}| \leq \frac{k\cos\alpha}{p(p+1)}$ which we shall prove in next section as theorem 5.5.1. Since a is arbitrary and $|a| < 1$ we can replace it by z^p.

$$\left|\frac{f''(z^p)}{f'(z^p)} - \frac{(1+e^{-2i\alpha})\bar{z}^p}{1-|z|^{2p}}\right| \leq \frac{k\cos\alpha}{1-|z|^{2p}}$$

or

$$(5.36) \qquad \left|z^{p-1}\ \frac{f''(z^p)}{f'(z^p)} - \frac{2e^{-i\alpha}\cos\alpha . \bar{z}|z|^{2(p-1)}}{1-|z|^{2p}}\right| \leq \frac{k|z|^{p-1}\cos\alpha}{1-|z|^{2p}}$$

From theorem 5.3.1 and the relation (5.36) if $g \varepsilon V_\alpha(k,p)$ then

$$(5.37) \qquad \left|\frac{g''(z)}{g'(z)} - \frac{2e^{-i\alpha}\cos\alpha\ \bar{z}\ \ z|^{2(p-1)}}{1-|z|^{2p}}\right| \leq \frac{k|z|^{p-1}\cos\alpha}{1-|z|^{2p}}$$

or

$$(5.38) \qquad \left|\frac{g''(z)}{g'(z)}\right| \leq \frac{2\cos\alpha\ |z|^{2p-1} + k\cos\alpha\ |z|^{p-1}}{1-|z|^{2p}} .$$

It is known that if

$$(5.39) \qquad \left|\frac{h_1''(z)}{h_1'(z)}\right| \leq \frac{\beta}{1-|z|^2} , \quad z \varepsilon D$$

then f(z) is univalent for some appropriate β . Robertson [58] has shown that β can be taken to be 1/2 while Becker [5] has shown that β can be taken atleast 1. If $h_2(z)$ is k-fold symmetric then it is obvious that if

$$(5.40)\qquad \left|\frac{h_2''(z)}{h_2'(z)}\right| \le \frac{\beta}{1-|z|^{2p}}$$

then $h_2(z)$ is univalent if $\beta = 1$. Using (5.38) and (5.40) we see that $g(z)$ is univalent if

$$2\cos\alpha \, . \, |z|^{2p-1} + k\cos\alpha \, . \, |z|^{p-1} \le 1.$$

This completes the proof of the theorem.

<u>Corollary 5.4.1</u> : <u>If</u> $g(z)$ <u>is</u> <u>in</u> $V_\alpha(k,p)$ <u>then</u>

$$(5.41)\qquad \log\left[\frac{(1-|z|^p)^{\frac{k-2}{2}}}{(1+|z|^p)^{\frac{k+2}{2}}}\right]^{\frac{\cos\alpha}{p}} \le \operatorname{Re}\{e^{i\alpha}\log g'(z)\} \le \log\left[\frac{(1+|z|^p)^{\frac{k-2}{2}}}{(1-|z|^p)^{\frac{k+2}{2}}}\right]^{\frac{\cos\alpha}{p}}.$$

<u>This result is sharp.</u>

<u>Proof</u> : From (5.37) we have

$$(5.42)\qquad \left|e^{i\alpha}\frac{zg''(z)}{g'(z)} - \frac{2\cos\alpha|z|^{2p}}{1-|z|^{2p}}\right| \le \frac{k\,|z|^p\cos\alpha}{1-|z|^{2p}}.$$

From this relation we get

$$\frac{2\cos|z|^{2p}-k\cos\alpha\,|z|^p}{1-|z|^{2p}} \le \operatorname{Re}\{e^{i\alpha}.\,re^{i\theta}.\,\frac{d}{d(re^{i\theta})}\log g'(re^{i\theta})\}$$

$$\le \frac{2\cos\alpha\,|z|^{2p}+k\cos\alpha\,|z|^p}{1-|z|^{2p}}$$

where $z = re^{i\theta}$.

Integrating this with respect to r we get the result. The bounds are sharp for all $k \geq 2$ and α, $|\alpha| < \pi/2$ as it can be seen from the function

$$(5.43) \qquad f'(z) = \left[\frac{(1+z^p)^{\frac{k}{2}-1}}{(1-z^p)^{\frac{k}{2}+1}}\right]^{\frac{e^{-i\alpha}\cos\alpha}{p}}, \quad f(0) = 0.$$

<u>Corollary 5.4.2</u> : <u>If</u> $g(z) \in V_\alpha(k,p)$ <u>then</u>

$$(5.44) \quad \frac{(1-|z|^p)^{\frac{\cos\alpha}{2p}(k-2\cos\alpha)}}{(1+|z|^p)^{\frac{\cos\alpha}{2p}(k+2\cos\alpha)}} \leq |g'(z)| \leq \frac{(1+|z|^p)^{\frac{\cos\alpha}{2p}(k-2\cos\alpha)}}{(1-|z|^p)^{\frac{\cos\alpha}{2p}(k+2\cos\alpha)}}.$$

<u>The result is sharp if</u> $\alpha = 0$.

<u>Proof</u> : From (5.37) we have

$$\left|\frac{zg''(z)}{g'(z)} - 2e^{-i\alpha}\frac{\cos\alpha\,|z|^{2p}}{1-|z|^{2p}}\right| \leq \frac{k\,|z|^p\cos\alpha}{1-|z|^{2p}}.$$

This can be rewritten as

$$(5.45) \quad \frac{2\cos^2\alpha\,.r^{2p} - k\cos\alpha\,r^p}{1-r^{2p}} \leq \mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} = r\frac{\partial}{\partial r}\log|f'(re^{i\theta})|$$

$$\leq \frac{2\cos^2\alpha\,.r^{2p} - k\cos\alpha . r^p}{1-r^{2p}}$$

where $z = re^{i\theta}$. Integrating this with respect to r we get the result. From function given in (5.43) it is easy to see that the result is sharp only if $\alpha = 0$.

Corollary 5.4.3. If $g(z)$ is in $V_\alpha(k,p)$ then $g(z)$ maps

$$(5.46) \qquad |z| < \left[\frac{2}{k\cos\alpha + \sqrt{k^2\cos^2\alpha - 4\cos 2\alpha}}\right]^{1/p}$$

onto a convex domain. This result is sharp only for $\alpha = 0$.

Proof : From (5.45) we have

$$(5.47) \qquad \mathrm{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} \geq \frac{(2\cos^2\alpha - 1)r^{2p} - k\cos\alpha .\, r^p + 1}{1 - r^{2p}}$$

$$= \frac{\cos 2\alpha .\, r^{2p} - k\cos\alpha\, r^p + 1}{1 - r^{2p}} .$$

Result follows from (5.47) and for $\alpha = 0$ sharpness follows from the function given in (5.43).

Theorem 5.4.4 : If $g(z)$ belongs to $V_\alpha(k,p)$ then $zf'(z)$ is α-spiral in the disc

$$(5.48) \qquad |z| < \left[\frac{k - \sqrt{k^2 - 4}}{2}\right]^{1/p} .$$

The result is sharp.

Proof : From (5.42) we get

$$\mathrm{Re}\left\{e^{i\alpha}\frac{zg''(z)}{g'(z)}\right\} \geq \frac{2\cos\alpha .\, r^{2p}}{1 - r^{2p}} - \frac{k\cos\alpha .\, r^p}{1 - r^{2p}}$$

or

$$\mathrm{Re}\left\{e^{i\alpha}\left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} \geq \frac{\cos\alpha\,(r^{2p} - k\, r^p + 1)}{1 - r^{2p}}$$

So

$$\mathrm{Re}\,\{e^{i\alpha} \cdot \frac{z[zg'(z)]'}{zg'(z)}\} > 0$$

if

$$r^{2p} - k\, r^{p} + 1 > 0$$

or

$$r < [\frac{k - \sqrt{k^2 - 4}}{2}]^{1/p} .$$

This result is sharp as it can be seen from the function in (5.43).

<u>Theorem 5.4.3.</u> <u>If $f(z)$ belongs to $V_o(k,p)$ then $f(z)$ is close-to-convex in the disc</u>

$$(5.49) \qquad |z| < [\sin \{\frac{p\pi}{2(k-2)}\}]^{1/p} .$$

<u>Proof</u> : From theorem 5.3.4 we have

$$f'(z) = \frac{[S_1(z)/z]^{\frac{k+2}{4}}}{[S_2(z)/z]^{\frac{k-2}{4}}}$$

where S_1 and S_2 are p-fold symmetric and starlike functions. This can be rewritten as

$$\frac{zf'(z)}{S_1(z)} = \frac{[S_1(z)/z]^{\frac{k-2}{4}}}{[S_2(z)/z]^{\frac{k-2}{4}}} .$$

Therefore

$$|\arg [\frac{zf'(z)}{f(z)}]| = |\arg \frac{[S_1(z)/z]^{\frac{k-2}{4}}}{[S_2(z)/z]^{\frac{k-2}{4}}}|$$

$$= \frac{k-2}{4} \cdot \frac{4\sin^{-1} r^p}{p} .$$

Here we have used the result proved in lemma 5.2.2. Now

$\text{Re}\ \{\frac{zf'(z)}{S_1(z)}\} > 0$ for $|z| < r$ if and only if $|\arg \frac{zf'(z)}{S_1(z)}| < \pi/2$ for $|z| < r$.

Thus $f(z)$ is close-to-convex (relative to the starlike function $S_1(z)$ if

$$\frac{k-2}{p} \sin^{-1} r^p < \pi/2,$$

and result follows from this relation.

<u>Theorem 5.4.4</u> : <u>If</u> $f(z)$ <u>belongs to</u> $V_\alpha(k,p)$ <u>then</u>

$$(5.50)\quad |\{f,z\}| = |[\frac{f''(z)}{f'(z)}]' - \frac{1}{2} [\frac{f''(z)}{f'(z)}]^2|$$

$$\leq k \cos \alpha . \frac{[(2+k \cos \alpha) r^p + 2p - 2}{2(1-r^p)^2} r^{p-2} .$$

<u>Proof</u> : We can assume $f(z)$ regular on $|z| = 1$ otherwise we can set $F_\rho(z) = \frac{f(\rho z)}{\rho}$ and let $\rho \to 1$ at the end of the proof. Since $f'(z) \neq 0$ in D we may use poisson formula to write

$$(5.51)\quad 1 + \frac{zf''(z)}{f'(z)} = \frac{e^{-i\alpha}}{2\pi} \int_0^{2\pi} \text{Re}\ \{e^{i\alpha} [1 + \frac{tf''(t)}{f'(t)}]\} \frac{t+z^p}{t-z^p} d\phi$$

This completes the proof of the theorem.

<u>Corollary 5.4.4</u> : <u>If</u> $f(z)$ <u>belongs to</u> $V_\alpha(k,p)$ <u>then</u> $f(z)$ <u>is univalent in</u> D <u>whenever</u>

(5.55) $$0 < \cos\alpha \leq \frac{\sqrt{p^2+1} - p}{k} .$$

<u>Proof</u> : From (5.50) we have

$$|\{f,z\}| \leq \frac{k \cos\alpha .\ r^{p-2}}{2(1-r^p)^2} [(2 + k\cos\alpha)\, r^p + 2p - 2]$$

$$= \frac{k\cos\alpha .\ (1+r^p)^2 r^{p-2}}{2(1-r^{2p})^2} [(2+k\cos\alpha)\, r^p + 2p - 2]$$

$$\leq \frac{2k\cos\alpha(2p + k\cos\alpha)}{(1-r^{2p})^2} .$$

By using Nehari's test [39] we see that $f(z)$ is univalent in D if

$$k\cos\alpha(2p + k\cos\alpha) \leq 1.$$

Result follows from this relation.

5.5 In this section we shall determine coefficient bounds for the functions in class $V_\alpha(k,p)$.

<u>Theorem 5.5.1</u> : <u>If</u> $f(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}$ <u>is in</u> $V_\alpha(k,p)$, <u>then</u>

(5.56) $$|a_{p+1}| \leq \frac{k\cos\alpha}{p(p+1)} .$$

<u>The bound is sharp for all</u> $k \geq 2$ <u>and all</u> α <u>such that</u> $|\alpha| < \pi/2$.

Proof : Differentiating logarithmically the representation of functions in $V_\alpha(k,p)$ obtained in the theorem 5.3.4, we have,

$$f''(z) = f'(z).\ \frac{e^{-i\alpha}\cos\alpha}{\pi}\int_0^{2\pi}\frac{z^{p-1}e^{i\theta}}{1-z^p e^{i\theta}}\,d\psi(\theta)$$

or

$$(5.57)\quad \frac{f''(z)}{z^{p-1}} = f'(z).\ \frac{e^{-i\alpha}\cos\alpha}{\pi}\int_0^{2\pi}\frac{e^{i\theta}}{1-z^p e^{i\theta}}\,d\psi(\theta).$$

If $z \to 0$ then from (5.57) we get

$$p(p+1)\ a_{p+1} = \frac{e^{-i\alpha}\cos\alpha}{\pi}\int_0^{2\pi} e^{i\theta}\,d\psi(\theta)$$

or

$$p(p+1)\ |a_{p+1}| \leq \frac{\cos\alpha}{\pi}\int_0^{2\pi}|d\psi(\theta)|$$

$$\leq k\cos\alpha\ .$$

Equality in (5.56) attains for the function given by

$$f'(z) = \left[\frac{(1+\epsilon z^p)^{\frac{k-2}{2}}}{(1-\epsilon z^p)^{\frac{k+2}{2}}}\right]^{\frac{e^{-i\alpha}\cos\alpha}{p}}\ .$$

<u>Theorem 5.5.2</u> : <u>If</u> $f(z)$ belon<u>gs</u> <u>to</u> $V_\alpha(k,p)$ <u>and</u> $f(z)=z+\sum_{n=1}^{\infty} a_{np+1}z^{np+1}$,

<u>then</u>

$$(5.58)\quad |a_{np+1}| \leq \frac{\prod_{m=0}^{n-1}\left(\frac{2}{p}+m\right)}{(np+1)\ \underline{n}}\left[\frac{k\cos\alpha+\sqrt{k^2\cos^2\alpha-4\cos 2\alpha}}{2}\right]^n .$$

Proof : Let

$$R = \left[\frac{2}{k\cos\alpha + \sqrt{k^2\cos^2\alpha - 4\cos 2\alpha}}\right]^{1/p}$$

be the radius of convexity for the class $V_\alpha(k,p)$, obtained in corollary 5.4.3. Then

$$\frac{f(Rz)}{R} = z + \sum_{n=1}^{\infty} a_{np+1} R^{np} z^{np+1}$$

$$\equiv z + \sum_{n=1}^{\infty} A_{np+1} z^{np+1}$$

is p-fold symmetric convex function in $|z| < 1$. It is well known that

$$|A_{np+1}| \leq \frac{\prod_{m=0}^{n-1}\left(\frac{2}{p}+m\right)}{(np+1)\,\underline{|n}} .$$

Therefore

$$|a_{np+1}| = \frac{|A_{np+1}|}{R^{np}}$$

$$\leq \frac{\prod_{m=0}^{n-1}\left(\frac{2}{p}+m\right)}{(np+1)\,\underline{|n}} \left[\frac{k\cos\alpha + \sqrt{k^2\cos^2\alpha - 4\cos\alpha}}{2}\right]^n .$$

Theorem 5.5. : <u>If</u> $f(z) = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}$ <u>belong to</u> $V_0(k,p)$ <u>then</u>

(5.59) $$|a_{2p+1}| \leq \frac{k^2+2p}{2p^2(2p+1)} \quad \text{if } k \geq 2p$$

<u>and</u>

(5.60) $$|a_{2p+1}| \leq \frac{2(2p+3\;k+4}{2p(2p+1)(4p-k+2)} \quad \text{if } 2 \leq k \leq 2p .$$

Proof : By lemma 5.2.4 we have

$$(5.61)\quad a_{2p+1} = \frac{1}{2p(2p+1)}\left[\frac{1}{\pi}\int_0^{2\pi} e^{-2pi\phi}\, d\mu(\phi) + \frac{(p+1)a_{p+1}}{\pi}\int_0^{2\pi} e^{-pi\phi}\, d\mu(\phi)\right]$$

and

$$(5.62)\quad a_{p+1} = \frac{1}{p(p+1)} \cdot \frac{1}{\pi}\int_0^{2\pi} e^{-pi\phi}\, d\mu(\phi).$$

From (5.61) and (5.62) we have

$$a_{2p+1} = \frac{1}{2p(2p+1)}\left[\frac{1}{\pi}\int_0^{2\pi} e^{-2pi\phi}\, d\mu(\phi) + \frac{1}{p}\left(\frac{1}{\pi}\int_0^{2\pi} e^{-pi\phi}\, d\mu(\phi)\right)^2\right].$$

We may assume without loss of generality that a_{2p+1} is real and non-negative since if not we consider $e^{i\theta}f(re^{i\theta}) = z + e^{pi\theta}a_{p+1}z^{p+1} + e^{2pi\theta}a_{2p+1}z^{2p+1} + \dots$ where θ is chosen such that $e^{2pi\theta}a_{2p+1}$ is real and non-negative. Thus we have

$$(5.63)\quad 2p(2p+1)a_{2p+1} = \frac{1}{\pi}\int_0^{2\pi} \cos 2p\phi\, d\mu(\phi) + \frac{1}{p}\left[\frac{1}{\pi}\int_0^{2\pi}\cos p\phi\, d\mu(\phi)\right]^2 - \frac{1}{p}\left[\frac{1}{\pi}\int_0^{2\pi}\sin p\phi\, d\mu(\phi)\right]^2$$

$$= \frac{1}{\pi}\int_0^{2\pi} 2\cos^2 p\phi\, d\mu(\phi) - 2 + \frac{1}{p}\left[\frac{1}{\pi}\int_0^{2\pi}\cos p\phi\, d\mu(\phi)\right]^2 - \frac{1}{p}\left[\frac{1}{\pi}\int_0^{2\pi}\sin p\phi\, d\mu(\phi)\right]^2$$

$$\leq \frac{1}{\pi}\int_0^{2\pi} 2\cos^2 p\phi\, d\mu(\phi) + \frac{1}{p}\left[\frac{1}{\pi}\int_0^{2\pi}\cos p\phi\, d\mu(\phi)\right]^2 .$$

Let us suppose that $\mu(\theta)$ is a step function with at most N jumps. Then if $\mu(\theta)$ has jumps $d_j\ \pi$ at θ_j $(0 \leq \theta_j \leq 2\pi)$,

$$(5.64) \qquad \sum_{j=1}^{N} d_j = 2\ , \quad \sum_{j=1}^{N} |d_j| \leq k.$$

From (5.63), (5.64) we have

$$(5.65) \quad 2p(2p+1)\ a_{2p+1} \leq 2 \sum_{j=1}^{N} \cos^2 p\theta_j \cdot d_j + \frac{1}{p}\Big[\sum_{j=1}^{N} \cos p\ \theta_j \cdot d_j\Big]^2 - 2\ .$$

Therefore we have to maximise the right hand side of (5.65) under the conditions in (5.64). There are two possibilities. In the first case $|\cos p\theta_j| = 1$ for all j $(1 \leq j \leq N)$ at the maximum. In this case from (5.65) we get

$$a_{2p+1} \cdot 2p(2p+1) \leq 2 \sum_{j=1}^{N} d_j + \frac{1}{p}\Big[\sum_{j=1}^{N} d_j\Big]^2 - 2$$

$$\leq 4 + \frac{1}{p} \cdot k^2 - 2$$

$$= \frac{k^2 + 2p}{p}$$

or

$$a_{2p+1} \leq \frac{k^2 + 2p}{2p^2(2p+1)} \quad .$$

In the second case let $|\cos p\ \theta_j| = 1$ not for all j. By renumbering, if necessary, let $|\cos p\ \theta_j| \neq 1$ for $1 \leq j \leq m$ where $m < N$. Then $\cos p\ \theta_j$, $1 \leq j \leq m$ are interior points of the interval $(-1,1)$. Since if maximum or minimum of some function is attained at some

interior point then derivative of the function is zero at that point, hence

$$\frac{d}{d \cos p\,\theta_n}\left[2\sum_{j=1}^{N} \cos^2 p\,\theta_j\, d_j + \frac{1}{p}\left\{\sum_{j=1}^{N} \cos p\,\theta_j \cdot d_j\right\}^2 - 2\right] = 0$$

if $1 \leq n \leq m$.

or

$$(5.66) \qquad 4 \cos p\,\theta_n\, d_n + \frac{2}{p}\left\{\sum_{j=1}^{N} \cos p\,\theta_j\, d_j\right\} d_n = 0$$

Hence $\cos p\,\theta_n$ is identically constant, say $\cos p\,\theta_n \equiv \cos p\beta$ for $1 \leq n \leq n$. Therefore, from (5.66) we have

$$(5.67) \qquad -4 \cos p\beta = \frac{2}{p}\sum_{j=1}^{N} \cos p\,\theta_j \cdot d_j$$

Substituting this in (5.65), we get,

$$(5.68) \quad 2p(2p+1)a_{2p+1} \leq 2\cos^2 p\beta \sum_{j=1}^{m} d_j + 2\sum_{j=m+1}^{N} d_j + 4p\cos^2 p\beta - 2 \;.$$

The conditions in (5.64) imply that

$$\sum_{j=1}^{m} = 2 - \sum_{n=m+1}^{N} d_j$$

and

$$\sum_{j=r+1}^{N} d_j \leq 1 + \frac{k}{2}\,.$$

Using these conditions, (5.68) can be rewritten as

$$2p(2p+1)\, a_{2p+1} \leq 2\cos^2 p\beta \,(2- \sum_{j=m+1}^{N} d_j) + 2 \sum_{j=m+1}^{N} d_j + 4p\cos^2 p\beta - 2$$

$$\leq 4(p+1)\cos^2 p\beta + 2(1-\cos^2 p\beta)\,(1+\tfrac{k}{2}) - 2$$

$$\leq k + \cos^2 p\beta\,(4p+2-k)$$

$$\leq \begin{cases} 4p+2 & \text{if } k \leq 4p+2 \\ k & \text{if } k \geq 4p+2 \end{cases}$$

$$\leq \frac{k^2+2p}{p} \qquad \text{if } k \geq 2p\ .$$

Let us now suppose that $k < 2p$. From (5.67) we have

$$-\cos p\beta\,(4 + \frac{2}{p}\sum_{j=1}^{m} d_j) = \frac{2}{p}\sum_{j=m+1}^{N} \cos p\theta_j\, d_j$$

and hence

$$|\cos p\beta| = \frac{2}{p}\left|\frac{\sum_{j=m+1}^{N} \cos p\,\theta_j\, d_j}{4 + \frac{2}{p}\sum_{j=1}^{m} d_j}\right|$$

$$\leq \frac{\frac{2}{p}(1+\frac{k}{2})}{4+\frac{2}{p}(1-\frac{k}{2})}$$

$$= \frac{(2+k)}{(4p+2-k)}\ .$$

Thus if $k < 2p$

$$2p(2p+1)\, a_{2p+1} \leq k + \left(\frac{2+k}{4p+2-k}\right)^2 \cdot (4p+2-k)$$

or

$$2p(2p+1)\, a_{2p+1} \leq \frac{2(2p+3)k + 4}{4p-k+2}\,.$$

Since step functions are dense in the family of functions of bounded variation with the normalization $\int_0^{2\pi} d\mu(\theta) = 2\pi$ and $\int_0^{2\pi} |d\mu(\theta)| \leq k\pi$ our results are valid for each function in $V_o(k,p)$.

Equality in (5.59) is attained for the function

$$f'(z) = \left[\frac{(1+\varepsilon z^p)^{\frac{k}{2}+1}}{(1-\varepsilon z^p)^{\frac{k}{2}-1}}\right]^{\frac{1}{p}}$$

and that in (5.60) for the function

$$f'(z) = \left[\frac{(1-z^p\, e^{-i\beta})^{\frac{1}{2}(\frac{k}{2}-1)}(1-z^p\, e^{+i\beta})^{\frac{1}{2}(\frac{k}{2}-1)}}{(1+z^p)^{\frac{1}{2}(1+\frac{k}{2})}}\right]^{1/p}$$

where

$$\beta = \frac{1}{2}\cos^{-1}\left(\frac{2+k}{4p-k+2}\right)$$

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